MATH 3030, Abstract Algebra
FALL 2012
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Homework Sheet 10

Due: Friday 25th January: 3:30 PM

Basic Questions

1. Which of the following are ideals?

   (i) The set of all polynomials whose constant term is 0 in $\mathbb{Q}[x]$.

   (ii) The set of all polynomials $a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ in $\mathbb{Z}[x]$ where a_1 is even.

   (iii) The set of pairs of the form $(0, b) \in \mathbb{Z} \times \mathbb{Z}$.

2. Which of the ideals in Q. 1 are

   (a) prime?

   (b) maximal?

3. What are the maximal ideals of $\mathbb{Z}_{24}$?

4. Describe all ring homomorphisms from $\mathbb{Z}$ to $\mathbb{Z}_{18}$.

5. Let $R = \mathbb{Z}_4 \times \mathbb{Z}_2$. Let I be the ideal of R generated by $(2, 1)$. What is the ring R/I?

Theoretical Questions

6. Let $\phi : R \longrightarrow S$ be a ring homomorphism.

   (a) Show that for an ideal I in R, the image $\phi(I)$ is an ideal in the image $\phi(R)$. Give an example to show that it need not be an ideal in S.

   (b) Show that for an ideal J in S, the inverse image $\phi^{-1}(J) = \{x \in R | \phi(x) \in J\}$ is an ideal in R.

7. Show that the intersection of a set of ideals in a ring R is another ideal in R.

8. Show that the composite of two ring homomorphisms is a ring homomorphism.

9. For a field F, show that any non-trivial proper prime ideal of $F[x]$ is maximal.

Bonus Questions

10. For ideals I and J of a ring R, show that $I + J = \{x + y | x \in I, y \in J\}$ is also an ideal of R.