MATH 3030, Abstract Algebra
FALL 2012
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Homework Sheet 11
Due: Monday 4th February: 3:30 PM

Basic Questions

1. Calculate the dimension of $Q[\sqrt[5]{7}]$ as a vector space over Q.
2. Give a basis of $Q[\frac{1}{2} + \frac{\sqrt{3}}{2}i]$ over Q.
3. What is $\mathrm{Irr}(\sqrt{3+\sqrt[3]{3}}, \mathbb{Q})$?
4. The polynomial $f(x) = x^2 + 2x + 2$ is irreducible over $\mathbb{Z}_3$. Let α be a zero of f, and factorise f over $\mathbb{Z}_3(\alpha)$. [Hint: use long division.]
5. Let α be a zero of $f(x) = x^3 + x + 1$ over $\mathbb{Z}_2$. Compute the multiplication table of $\mathbb{Z}_2(\alpha)$. [Hint: $\mathbb{Z}_2(\alpha)$ has 8 elements: 0, 1, α, $\alpha + 1$, α^2, $\alpha^2 + 1$, $\alpha^2 + \alpha$, and $\alpha^2 + \alpha + 1$.]

Theoretical Questions

6. Let V be a vector space of dimension n over a field F.

   (a) Show that if $v_1, \ldots, v_n$ is a linearly independent set, then it is a basis.

   (b) Show that if $v_1, \ldots, v_n$ is a spanning set, then it is a basis.
7. If F is a finite field with q elements, and V is a vector space of dimension d over F, show that V has q^d elements.
8. Show that if E is a finite extension field of F, and if $[E : F]$ is prime, then E is a simple extension of F. [Hint: in fact $E = F(\alpha)$ for any α in $E \setminus F$.]
9. Let F be a field, let $F(\alpha)$ be algebraic over F, and let $[F(\alpha) : F]$ be odd. Show that $F(\alpha^2) = F(\alpha)$.