MATH 3030, Abstract Algebra
FALL 2012
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Homework Sheet 14
Due: Friday 15th March: 3:30 PM

Basic Questions

1. Which of the following pairs of numbers are conjugate over $\mathbb{Q}$?

   (a) $\sqrt{2}$ and $\sqrt{6}$.

   (b) $1+\sqrt{2}$ and $1-\sqrt{2}$.

   (c) $\sqrt[4]{2}$ and $\sqrt{2}$.

2. In $\mathbb{Q}(\sqrt{2}+\sqrt{3})$, compute $\psi_{\sqrt{2}+\sqrt{3},\sqrt{2}-\sqrt{3}}(2+\sqrt{2}-\sqrt{6})$.

3. In $\mathbb{Q}(\sqrt{2}+\sqrt{3})$, compute the fixed field of $\{\psi_{\sqrt{2}+\sqrt{3},-\sqrt{2}-\sqrt{3}}\}$.

4. Let α be a zero of x^3+x^2+x+3 in GF(125).

   (a) Compute the Frobenius automorphism $\sigma_5(\alpha)$. [Express $\sigma_5(\alpha)$ in the basis $\{1,\alpha,\alpha^2\}$.]

   (b) Describe the fixed field of $\{\sigma_5\}$ in terms of this basis.

5. Let $\omega=\frac{-1+\sqrt{3}i}{2}$ (so that $\omega^3=1$.) Consider the isomorphism $\psi_{\sqrt[3]{2},\omega\sqrt[3]{2}}$ from $\mathbb{Q}\left(\sqrt[3]{2}\right)$ to $\mathbb{Q}\left(\sqrt[3]{2}\omega\right)$. Compute all ways to extend this isomorphism to an isomorphism mapping $\mathbb{Q}(\sqrt[3]{2},\omega\sqrt[3]{2})$ to a subfield of $\overline{Q}$.

Theoretical Questions

6. Let $F(\alpha_1,\ldots,\alpha_n)$ be an extension field of F. Show that any automorphism σ of $F(\alpha_1,\ldots,\alpha_n)$ leaving F fixed is completely determined by the values $\sigma(\alpha_i)$.

7. Let E be an extension field of F. Let S be a set of automorphisms of E fixing F. Let H be the subgroup of $G(E/F)$ generated by S. Show that $E_S=E_H$.

8. (a) Show that if F is an algebraically closed field, then any isomorphism σ of F to a subfield of F such that F is algebraic over $\sigma(F)$, is an automorphism of F. [Hint, since $\sigma(F)$ is isomorphic to F, it must be algebraically closed.]

   (b) Let E be an algebraic extension of F. Show that any isomorphism of E onto a subfield of $\overline{F}$ that fixes F can be extended to an automorphism of $\overline{F}$.

9. Let E be an algebraic extension of F. Show that there is an isomorphism of $\overline{F}$ to $\overline{E}$ fixing all elements of F.

10. Let E be a finite extension of F. Show that $\{E : F\} \leqslant [E : F]$. [You may assume the result for simple extensions.]

Bonus Questions

11. Show that if α and β are both transcendental over F, then there is an isomorphism of $F(\alpha)$ and $F(\beta)$ sending α to β.

12. Show that the only automorphism of $\mathbb{R}$ is the identity. [Hint: show that any automorphism preserves positive numbers (since these are the squares of real numbers) and therefore preserves the order on real numbers.]