MATH 3030, Abstract Algebra
FALL 2012
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Homework Sheet 3
Due: Friday 12th October: 3:30 PM

Basic Questions

1. (a) Calculate the product $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 5 & 1 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 5 & 4 & 1 \end{pmatrix}$.

   (b) Calculate the inverse of $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 1 & 2 & 5 \end{pmatrix}$.

2. Write $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 3 & 7 & 9 & 5 & 8 & 1 & 4 & 6 \end{pmatrix}$ as a product of disjoint cycles.

3. How many permutations $\sigma \in S_6$ satisfy $\sigma^2 = e$?

4. Recall that the order of an element x is the smallest power $n \geqslant 1$ such that $x^n = e$.

   (a) What is the order of $(125)(34)$?

   (b) What is the order of $(1467)(35)$?

5. What is the largest order of an element of S_9?

Theoretical Questions

6. A permutation group $H \leqslant S_A$ on a set A is *transitive* if for any two elements $a, b \in A$, there is a permutation $\sigma \in H$ such that $\sigma(a) = b$. Show that a transitive permutation group must have at least $|A|$ elements.

7. Let $B \subseteq A$. Show that the set of permutations of A that fix B, i.e. the set $\{\sigma \in S_A | (\forall b \in B)(\sigma(b) \in B)\}$ is a subgroup of S_A.

8. (a) Show that S_n is generated by the transpositions $(1,2), (2,3), \ldots, (n-1, n)$.

   (b) Show that S_n is generated by just the two elements $(1,2)$ and $(1,2,3,\ldots,n)$.

9. Show that any subgroup of S_n which is cyclic and transitive must have order n.

10. Show that the set of 3-cycles generates the alternating group A_n.

11. Show that permutations σ and τ are conjugate in S_n [that is, there is a permutation θ such that $\tau = \theta\sigma\theta^{-1}$] if and only if they have the same cycle type (that is, they have the same number of cycles, and the corresponding cycles have the same size).

Bonus Questions

12. If G is a permutation group on a set X, and $x \in X$, the *stabiliser* of x is the set of elements of G which fix x. That is $\sigma_G(x) = \{g \in G | g(x) = x\}$. Show that $|G| = |O_G(x)||\sigma_G(x|$ where $O_G(x)$ is the orbit of x under G.