# MATH 3030, Abstract Algebra
## FALL 2012
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### Midterm Examination
### Model Solutions

## Basic Questions

1. *Which of the following are groups:*

   *(a) The set of functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(1) = 0$ with pointwise addition (i. e. $(f+g)(x) = f(x)+g(x)$).*

   This is a group. Pointwise addition is well-defined on this set, and is clearly associative. The constantly zero function is the identity, and the inverse function of $f$ is $g$ given by $g(x) = -f(x)$.

   *(b) The set of functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(1) = 4$ with pointwise addition (i. e. $(f+g)(x) = f(x)+g(x)$).*

   This is not a group since the operation is not well-defined on the set — if we add two functions $f$ and $g$ in this set, we will have $(f+g)(1) = 4+4 = 8$, so the pointwise sum is not in the set.

2. *How many generators are there in the cyclic group $\mathbb{Z}_{36}$?*

   Generators of this group are numbers that are coprime to 36. That is, the generators are $\{1, 5, 7, 11, 13, 17, 19, 23, 25, 29, 31, 35\}$, so there are 12 generators.

3. *Which of the following are subgroups of $\mathbb{Z} \times \mathbb{Z}$?*

   *(a) The set of all pairs $(a, b)$ such that $a^2 + b^2$ is a square number (i.e. $a^2 + b^2 = c^2$ for some $c \in \mathbb{Z}$.)*

   This is not a subgroup because it is not closed. For example it contains $(3, 4)$ and $(4, 3)$ but not $(3, 4) + (4, 3) = (7, 7)$.

   *(b) The set of all pairs $(a, b)$ such that $a \geqslant b$.*

   This is not a subgroup because it is not closed under inverses. For example it contains $(3, 1)$ but not $(-3, -1)$.

4. *(a) Write $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 5 & 4 & 3 & 7 & 8 & 1 & 6 & 9 \end{pmatrix}$ as a product of disjoint cycles.*

   $\sigma = (1257)(34)(68)$.

   *(b) What is the order of $\sigma$?*

   The order of $\sigma$ is the least common multiple of its cycle lengths, which is 4.

*(c) Which of the following permutations are conjugate to $\sigma$ in $S_9$?*

*(i)* $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 5 & 9 & 3 & 7 & 8 & 1 & 6 & 4 \end{pmatrix}$

*(ii)* $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 5 & 2 & 7 & 9 & 1 & 8 & 4 & 6 & 3 \end{pmatrix}$

*(iii)* $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 3 & 2 & 1 & 5 & 6 & 8 & 9 & 7 \end{pmatrix}$

A permutation is conjugate to $\sigma$ if and only if it is of the same cycle type. The above permutations have the following representations as products of disjoint cycles:

(i) (1257)(394)(68)

(ii) (15)(3749)(68)

(iii) (14)(23)(789)

so only (ii) is conjugate to $\sigma$.

5. *Is the subgroup of $S_5$ generated by* (12)(345) *and* (35) *normal?*

   This subgroup is the group of permutations that fix the set $\{1, 2\}$. This is not normal, since conjugation by (13) for example does not fix this group.

6. *Consider the function $f : GL(3, \mathbb{R}) \to \mathbb{R}$, where $GL(3, \mathbb{R})$ is the group of $3 \times 3$ invertible matrices under matrix multiplication, and $f$ is the function sending a matrix to its trace (the sum of the diagonal elements). Is $f$ a homomorphism?*

   This is not a homomorphism, since for example, the trace of the identity is 3, and $3 \times 3 \neq 3$.

7. *Calculate the centre of $D_4 \times D_6$.*

   Elements $(a, x)$ and $(b, y)$ of $D_4 \times D_6$ commute if and only if $a$ and $b$ commute and $x$ and $y$ commute, so the centre of $D_4 \times D_6$ consists of pairs of the form $(x, y)$ where $x$ is in the centre of $D_4$ and $y$ is in the centre of $D_6$. Both these centres consist of the identity and rotation by 180°, so the centre of $D_4 \times D_6$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

# Theoretical Questions

8. *Prove that the intersection of two subgroups of a group is another subgroup.*

   Let $H$ and $K$ be subgroups of a group $G$. We want to show that $H \cap K$ is a subgroup of $G$.

   - If $x, y \in H \cap K$, then $x, y \in H$, so $xy \in H$ since $H$ is a subgroup, and also $x, y \in K$, so $xy \in K$ since $K$ is a subgroup. Therefore, $xy \in H \cap K$.

- We have $e \in H$ and $e \in K$, so $e \in H \cap K$.
- If $x \in H \cap K$, then $x^{-1} \in H$ and $x^{-1} \in K$, so $x^{-1} \in H \cap K$.

9. *State and prove Lagrange's theorem about the order of a subgroup of a finite group.*

**Theorem 1** (Lagrange)**.** *If $G$ is a finite group, and $H$ is a subgroup of $G$, then $|H|$ divides $|G|$.*

*Proof.* Consider the cosets $xH$ for elements $x \in G$. These form a partition of $G$. Each of them has $|H|$ elements, and $G$ is the disjoint union of these cosets, so $|G|$ is a sum of copies of $|H|$, so it is divisible by $|H|$. □

10. *Let $H$ be an abelian normal subgroup of $G$. Show that the subgroup generated by $H$ and the centre $Z(G)$ is also abelian and normal.*

Let $K$ be the subgroup generated by $H$ and $Z(G)$. Clearly any two generators of $K$ commute, since either one of them is in $Z(G)$ and therefore commutes with any element of $G$, or they are both in $H$ and commute because $H$ is abelian. Therefore, $K$ is abelian. We want to show that $K$ is normal. Let $x \in G$. Since conjugation by $x$ is an automorphism of $G$ (an isomorphism from $G$ to itself), we have that $xKx^{-1}$ is a subgroup of $G$, generated by $xHx^{-1} = H$ and $xZ(G)x^{-1} = Z(G)$. Therefore $xKx^{-1} = K$, so since $x$ was arbitrary, $K$ is a normal subgroup of $G$.