MATH 3030, Abstract Algebra
FALL 2012
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Homework Sheet 10
Model Solutions

Basic Questions

1. *Which of the following are ideals?*

   (i) The set of all polynomials whose constant term is 0 in $\mathbb{Q}[x]$.

   Let f and g be polynomials in $\mathbb{Q}[x]$ with constant terms 0. Clearly the constant term in $f+g$ is also 0. Let h be any polynomial in $\mathbb{Q}[x]$. Clearly the constant term of fh is also 0, so this is an ideal. (It is the ideal generated by x.)

   (ii) The set of all polynomials $a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ *in* $\mathbb{Z}[x]$ *where* a_1 *is even.*

   This is not an ideal. It is a subring, but for example, $f(x) = 2x + 1$ is in the set, and if $g(x) = x$, then the product $fg(x) = 2x^2 + x$ is not in the set.

   (iii) The set of pairs of the form $(0, b) \in \mathbb{Z} \times \mathbb{Z}$.

   This is an ideal. If we add two such pairs $(0, b)$ and $(0, b')$, we get another such pair $(0, b+b')$. If (x, y) is any pair in $\mathbb{Z}\times\mathbb{Z}$, then $(x, y)(0, b) = (0, by)$ which is in the ideal.

2. *Which of the ideals in Q. 1 are*

   (a) prime?

   Both ideals (i) and (iii) are prime. The constant term of a product of polynomials is the product of their constant terms, so if it is zero, then one of the constant terms must be zero (since $\mathbb{Q}$ is an integral domain).

   In (iii), if we have $(k, l)(m, n) = (0, b)$, then we must have $km = 0$, so that either $k = 0$ or $m = 0$.

   (b) maximal?

   The ideal I in (i) is maximal, since if J is a larger ideal, we can let $f(x) = a_0 + a_1x + \cdots + a_nx^n \in J \setminus I$, so that $a_0 \neq 0$. Now $g(x) = a_1x + \cdots + a_nx^n$ is in I, so is also in J. Therefore, $(g-f)(x) = a_0$ is in J. Since $\mathbb{Q}$ is a field, this means that any constant polynomial is in J, and so all polynomials are in J. (The quotient of $\mathbb{Q}[x]$ by I is isomorphic to $\mathbb{Q}$.)

   The ideal in (iii) is not maximal, since it is contained in the ideal of pairs of the form $(0, 2b)$. (The quotient of $\mathbb{Z} \times \mathbb{Z}$ by this ideal is isomorphic to $\mathbb{Z}$.)

3. *What are the maximal ideals of $\mathbb{Z}_{24}$?*

   The ideals of $\mathbb{Z}_{24}$ are sets of all multiples of n for elements n that divide 24. The maximal ideals correspond to the n that are prime, namely $n = 2$ and $n = 3$. That is the maximal ideals are $\{2x | x \in \mathbb{Z}_{24}\}$ and $\{3x | x \in \mathbb{Z}_{24}\}$.

4. *Describe all ring homomorphisms from $\mathbb{Z}$ to $\mathbb{Z}_{18}$.*

   A ring homomorphism f from $\mathbb{Z}$ to $\mathbb{Z}_{18}$ is entirely determined by $f(1)$. Since $1^2 = 1$, we must have $f(1)^2 = f(1)$, so $f(1)$ must be a solution to $x^2 - x = 0$. The solutions to this in $\mathbb{Z}_{18}$ are 0, 1, 9 and 10, so the homomorphisms from $\mathbb{Z}$ to $\mathbb{Z}_{18}$ are given by $f(1) = 0$ (the trivial homomorphism), $f(1) = 1$, $f(1) = 9$ and $f(1) = 10$.

5. *Let $R = \mathbb{Z}_4 \times \mathbb{Z}_2$. Let I be the ideal of R generated by $(2, 1)$. What is the ring R/I?*

   The ideal generated by $(2, 1)$ consists of all pairs $(a, b) \in \mathbb{Z}_4 \times \mathbb{Z}_2$ such that a is even. This ideal has 4 elements, while R has 8, so the factor ring has 2 elements, so it must be isomorphic to $\mathbb{Z}_2$.

Theoretical Questions

6. *Let $\phi : R \longrightarrow S$ be a ring homomorphism.*

   (a) Show that for an ideal I in R, the image $\phi(I)$ is an ideal in the image $\phi(R)$. Give an example to show that it need not be an ideal in S.

   We already know that the image $\phi(I)$ is a subring of $\phi(R)$, so we just need to check that it is closed under multiplication by arbitrary elements of $\phi(R)$. Let $x \in \phi(I)$ and let $y \in \phi(R)$. We have that $x = \phi(a)$ for some $a \in I$ and $y = \phi(b)$ for some $b \in R$. Therefore we have that $xy = \phi(a)\phi(b) = \phi(ab)$. However, since I is an ideal in R, we know that $ab \in I$, so that $\phi(ab) \in \phi(I)$ as required. The argument that $yx \in \phi(I)$ is similar. Finally, $-x = \phi(-a) \in \phi(I)$.

   (b) Show that for an ideal J in S, the inverse image $\phi^{-1}(J) = \{x \in R | \phi(x) \in J\}$ is an ideal in R.

   Let $x, y \in \phi^{-1}(J)$, and $z \in R$. We have that $\phi(x + y) = \phi(x) + \phi(y) \in J$, so $x + y \in \phi^{-1}(J)$, and $\phi(-x) = -\phi(x) \in J$, so $-x \in \phi^{-1}(J)$. We also have that $\phi(xz) = \phi(x)\phi(z) \in J$, so $xz \in \phi^{-1}(J)$. Similarly, $zx \in \phi^{-1}(J)$. Therefore $\phi^{-1}(J)$ is an ideal in R.

7. *Show that the intersection of a set of ideals in a ring R is another ideal in R.*

   Let I_i be a set of ideals in R, and let J be their intersection. For $x, y \in J$, and $z \in R$, we have that for any i, $x \in I_i$ and $y \in I_i$, so that $-x$, $x + y$, xz and zx are all in I_i. Therefore, $-x$, $x + y$, xz and zx are all in J, so J is an ideal.

8. *Show that the composite of two ring homomorphisms is a ring homomorphism.*

   Let $f : R \longrightarrow S$ and $g : S \longrightarrow T$ be two ring homomorphisms. We need to show that the composite $gf : R \longrightarrow T$ is a ring homomorphism. That is, we need to show that for any $x, y \in R$, $gf(x + y) = gf(x) + gf(y)$ and $gf(xy) = gf(x)gf(y)$. Now we have $gf(x + y) = g(f(x) + f(y)) = gf(x) + gf(y)$ and $gf(xy) = g(f(x)f(y)) = gf(x)gf(y)$.

9. *For a field F, show that any non-trivial proper prime ideal of $F[x]$ is maximal.*

   Let I be a nontrivial prime ideal of $F[x]$. Since I is a principal ideal generated by f for some $f \in F[x]$. We know that I is maximal if and only if f is irreducible. We need to show that if I is prime, then f is irreducible. However, if I is prime, then if f factors as $f = gh$, we must have $g \in I$ or $h \in I$. Since f generates I, this means that $g = fk$ or $h = fk$. This forces g or h to be constant polynomials, so f is irreducible.

Bonus Questions

10. *For ideals I and J of a ring R, show that $I + J = \{x + y | x \in I, y \in J\}$ is also an ideal of R.*

    Let $a, b \in I + J$ and let $c \in R$. We have that $a = a_I + a_J$ and $b = b_I + b_J$ for some $a_I, b_I \in I$ and $a_J, b_J \in J$. Therefore, we have $a + b = a_I + a_J + b_I + b_J = (a_I + b_I) + (a_J + b_J) \in I + J$. Also, $-a = -a_I - a_J \in I + J$ and $ca = c(a_I + a_J) = ca_I + ca_J \in I + J$ and $ac = (a_I + a_J)c = a_I c + a_J c \in I + J$, so $I + J$ is an ideal.