MATH 3030, Abstract Algebra
FALL 2012
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Homework Sheet 3
Due: Wednesday 10th October: 3:30 PM

Basic Questions

1. *(a) Calculate the product* $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 5 & 1 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 5 & 4 & 1 \end{pmatrix}$.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 4 & 1 & 2 \end{pmatrix}$$

(b) Calculate the inverse of $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 1 & 2 & 5 \end{pmatrix}$.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 2 & 1 & 5 \end{pmatrix}$$

2. *Write* $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 3 & 7 & 9 & 5 & 8 & 1 & 4 & 6 \end{pmatrix}$ *as a product of disjoint cycles.*

$$(1,2,3,7)(4,9,6,8)$$

3. *How many permutations* $\sigma \in S_6$ *satisfy* $\sigma^2 = e$*?*

An element $\sigma \in S_6$ satisfies $\sigma^2 = e$ if and only if σ is a product of disjoint 2-cycles. There is one permutation which is a product of 0 2-cycles (the identity). There are $\binom{6}{2} = 15$ permutations which are a product of 1 2-cycle. There are $\frac{1}{2}\binom{6}{2,2,2} = 45$ permutations which are a product of 2 2-cycle, and there are $\frac{1}{3!}\binom{6}{2,2,2} = 15$ permutations which are a product of 3 disjoint 2-cycles. That is a total of 76 permutations.

4. *Recall that the order of an element* x *is the smallest power* $n \geqslant 1$ *such that* $x^n = e$.

(a) What is the order of $(125)(34)$*?*

6. The powers are: $(125)(34)$, (152), (34), (125) and $(152)(34)$.

(b) What is the order of $(1467)(35)$*?*

4. The powers are $(1467)(35)$, $(16)(47)$ and $(1764)(35)$.

5. *What is the largest order of an element of* S_9*?*

The order of an element which is a product of disjoint cycles of lengths $i_1, i_2, \ldots, i_n$ is the least common multiple of $i_1, i_2, \ldots, i_n$. We are therefore looking for a combination of numbers whose sum is 9, with the largest least common multiple. If we have 3 numbers that sum to 9, one must be a multiple of another, so we do better to choose 2 numbers. The best we can do is 4 and 5, making an element of order 20. For example (1234)(56789).

Standard Questions

6. *A permutation group $H \leqslant S_A$ on a set A is* transitive *if for any two elements $a, b \in A$, there is a permutation $\sigma \in H$ such that $\sigma(a) = b$. Show that a transitive permutation group must have at least $|A|$ elements.*

   By fixing a, and letting b vary over the elements of A, we see that for each of the $|A|$ choices for b, there is a permutation $\sigma_b \in H$, with $\sigma_b(a) = b$. These σ_b must all be distinct, so there must be at least $|A|$ elements in H.

   [The example of the subgroup generated by a cyclic permutation on A shows that $|A|$ elements is a possibility.]

7. *Let $B \subseteq A$, where A is finite. Show that the set of permutations of A that fix B, i.e. the set $\{\sigma \in S_A | (\forall b \in B)(\sigma(b) \in B)\}$ is a subgroup of S_A.*

   Let σ and τ be two permutations that fix B. We need to show that $\sigma\tau$ and σ^{-1} fix B. For $b \in B$, we have that $\tau(b) \in B$, so that $\sigma(\tau(b)) \in B$, so that $\sigma\tau$ fixes B. Because σ is a bijection, we must have that $\{\sigma(b) | b \in B\} = B$. Now for $b \in B$, we must have that $b = \sigma(x)$ for some $x \in B$. Therefore $\sigma^{-1}(b) = x \in B$.

8. *(a) Show that S_n is generated by the transpositions $(1, 2), (2, 3), \ldots, (n - 1, n)$.*

   We know that the set of all transpositions generates S_n, so it is sufficient to show that any transposition is a product of adjacent transpositions. Consider the transpositon (i, j) where $i < j$. We can express it as the product

   $$(i, i+1)(i+1, i+2) \cdots (j-2, j-1)(j-1, j)(j-2, j-1) \cdots (i+1, i+2)(i, i+1)$$

   (b) Show that for $n \geqslant 2$, S_n is generated by just the two elements $(1, 2)$ and $(1, 2, 3, \ldots, n)$.

   By part (a), it is sufficient to express the transpositions of the form $(i, i + 1)$ as products of $(1, 2)$ and $(1, 2, 3, \ldots, n)$. However, we see that $(1, 2, 3, \ldots, n)^i(1, 2)(1, 2, 3, \ldots, n)^{-i} = (i + 1, i + 2)$, so we can express all such transpositions.

9. *Show that any subgroup of S_n which is cyclic and transitive must have order n.*

   Let $< \sigma >$ be a cyclic transitive subgroup of S_n. Since σ is transitive, for any $i \in \{1, \ldots, n\}$, we have that the orbit of i under x contains the whole of $\{1, \ldots, n\}$. That is, x must consist of a single cycle. It must therefore have order n.

10. *Show that the set of 3-cycles generates the alternating group A_n.*

We need to show that any even permutation is a product of 3-cycles. An even permutation is one which has an even number of cycles of even length. We therefore need to show that any cycle of odd length is a product of 3-cycles, and that any product of two disjoint cycles of even length is a product of 3-cycles.

For cycles of odd length, we consider the example $(1, 2, \ldots, 2k+1)$. All other cycles are conjugate to this example, so it is sufficient to express this cycle as a product of 3-cycles. We proceed by induction on k. $k = 1$ is a 3-cycle. Now for $k > 1$, we have $(1, 2, \ldots, 2k+1) = (1, 2k, 2k+1)(1, 2, \ldots, 2k-1)$, so by induction, we can express $(1, 2, \ldots, 2k+1)$ as a product of 3-cycles.

For the product of two disjoint cycles of even length, it is sufficient to prove the case of two disjoint transpositions, since we already know that each of the disjoint cycles can be expressed as a product of an odd number of transpositions, so we can choose representations of each cycle as products of transpositions; we can extend the shorter product by a collection of inverse pairs (this is possible, since both products have odd length); this gives the pair of disjoint cycles as a product of pairs of disjoint transpositions.

Now we have that $(1, 2)(3, 4) = (1, 4, 3)(1, 2, 3)$, and by conjugation, we can express any product of two disjoint transpositions as a product of 3-cycles.

11. *Show that permutations σ and τ are conjugate in S_n [that is, there is a permutation θ such that $\tau = \theta\sigma\theta^{-1}$] if and only if they have the same cycle type (that is, they have the same number of cycles, and the corresponding cycles have the same size).*

Let $\tau = \theta\sigma\theta^{-1}$. For any element x we have that $\tau(\theta(x)) = \theta(\sigma(\theta^{-1}(\theta(x)))) = \theta(\sigma(x))$. Similarly, we get that $tau^k(\theta(x)) = \theta(\sigma^k(x))$ for any k. Therefore, the orbits of τ are just the images of the orbits of σ. Thus σ and τ must have the same cycle type.

Conversely, suppose that σ and τ have the same cycle type. We want to construct θ such that $\tau = \theta\sigma\theta^{-1}$. We form a bijection between the cycles of σ and the cycles of τ, such that each cycle is paired with one of the same length. Now we pick one element from each cycle of σ and one element from each cycle of τ. Now we want to construct a permutation θ such that $\tau = \theta\sigma\theta^{-1}$. Let $x_1, x_2, \ldots, x_k$ be a cycle of σ and $y_1, y_2, \ldots, y_k$ be the corresponding cycle of τ. Now if we define $\theta(x_i) = y_i$, then we have that $\theta\sigma\theta^{-1}(y_i) = \theta\sigma(x_i) = \theta(x_{i+1}) = y_{i+1} = \tau(y_i)$, so that τ and $\theta\sigma\theta^{-1}$ agree on the elements of the cycle $y_1, y_2, \ldots, y_k$. Defining θ in a similar way for all cycles, we get that τ and σ are conjugate.

Bonus Questions

12. *If G is a permutation group on a set X, and $x \in X$, the* stabiliser *of x is the set of elements of G which fix x. That is $\sigma_G(x) = \{g \in G | g(x) = x\}$. Show that $|G| = |O_G(x)||\sigma_G(x)|$ where $O_G(x)$ is the orbit of x under G.*

    For each element y in $O_G(x)$, we can choose an element $g_y \in G$ such that $g_y(x) = y$. Now for any $h \in G$ such that $h(x) = y$, we have ${g_y}^{-1}h(x) = {g_y}^{-1}(y) = x$, so that ${g_y}^{-1}h \in \sigma_G(x)$. That is, the sets $\{g \in G | g(x) = y\}$ are all in bijection with $\sigma_G(x)$, and G is partitioned into $|O_G(x)|$ such sets, so we get $|G| = |O_G(x)||\sigma_G(x)|$. [The sets $\{g \in G | g(x) = y\}$ are the cosets of the subgroup $\sigma_G(x)$. This is therefore a special case of Lagrange's theorem.]