MATH/STAT 3360, Probability
FALL 2011
Toby Kenney
Sample Midterm Examination
Model Solutions

This Sample Midterm has more questions than the actual midterm, in order to cover a wider range of questions.

1. *How many ways are there to choose 5 students from a class of 25?*

   There are $\binom{25}{5}$ ways. [$= \frac{25\times24\times23\times22\times21}{5!} = 53130$.]

2. *There are n people and m books in a library, where $n < m$. Each person selects one book to read. In how many different ways can this be achieved?*

   If we number the n people from 1 to n, then each way in which this can be achieved corresponds to an ordered list of n of the books. The number of such lists is ${}_mP_n = m(m-1)\cdots(m+1-n)$.

3. *How many distinct ways can the letters of the word "PERMUTATION" be arranged?*

   "PERMUTATION" has 11 letters. Two are "T" and the others are all distinct, so the number of distinct arangements is $\frac{11!}{2!}$. [=19958400]

4. *What is the probability that the sum of 3 fair 6-sided dice is 6?*

   Three 6-sided dice can sum to 6 in the following ways:

   (1,1,4), (1,4,1), (4,1,1), (1,2,3), (1,3,2), (2,1,3), (2,3,1), (3,1,2), (3,2,1), (2,2,2). That is a total of 10 ways, so the probability is $\frac{10}{6^3}$.

5. *What is the probability that a five-card poker hand is a four of a kind (has four cards of one rank and one card of another)?*

   The number of four of a kind hands is 13×48. (There are 13 possible ranks for the four of a kind, and this completely determines those cards, and then there are 48 possible choices for the last card.) The total number of hands is $\binom{52}{5}$, so the probability of a four of a kind is $\frac{13\times48}{\binom{52}{5}}$. [$= \frac{5!}{4\times51\times50\times49} = \frac{1}{17\times5\times49} = \frac{1}{4165}$.]

6. *A fair coin is tossed 7 times.*

   (a) What is the probability that the sequence HHHT occurs somewhere in the 7 tosses?

   Let E_1, E_2, E_3 and E_4 be the events that the sequence HHHT occurs starting on toss number 1, 2, 3 and 4 respectively. These events are mutually exclusive, and each has probability $\frac{1}{16}$, so the probability that one of them occurs is $\frac{4}{16} = \frac{1}{4}$.

(b) What is the probability that the sequence THTH occurs somewhere in the 7 tosses?

This is the union of 4 events, namely a sequence of THTH starting on toss i for $i = 1, 2, 3, 4$. Let these events be S_1, S_2, S_3 and S_4. Each has probability $\frac{1}{16}$, and furthermore, $P(S_1 \cap S_3) = \frac{1}{64}$ and $P(S_2 \cap S_4) = \frac{1}{64}$, and it is not possible for three or more of the events to happen at once. Therefore, the probability that THTH occurs somewhere is $4 \times \frac{1}{16} - 2 \times \frac{1}{64} = \frac{7}{32}$.

7. *five coins are tossed: are the following events independent?*

   (i) There are the same number of heads among the first two tosses and the last two tosses.

   (ii) The total number of heads is 3.

   Let A be the event described in (i) and B be the event described in (ii). Now the outcomes in $A \cap B$ are HTHHT, HTHTH, THHHT and THHTH, so $P(A \cap B) = \frac{4}{32} = \frac{1}{8}$. The number of outcomes in B is $\binom{5}{3} = 10$, so $P(B) = \frac{5}{16}$. Therefore, in order for the events to be independent, we would need $P(A) = \frac{P(A \cap B)}{P(B)} = \frac{2}{5}$, but this is clearly impossible, since the probability of A must be $\frac{n}{32}$ for some n, and $\frac{2}{5}$ is not of this form. Alternatively, we can explicitly count the outcomes in A. They are TTTTT, TTHTT, HTTHT, HTTTH, THTHT, THTTH, HTHHT, HTHTH, THHHT, THHTH, HHTHH, and HHHHH. Therefore $P(A) = \frac{12}{32} = \frac{3}{8}$, and $P(A)P(B) = \frac{3}{8} \times \frac{5}{16} = \frac{15}{64} \neq \frac{1}{8}$.

8. *Suppose the number of cars that want to park in a particular street each day is a Poisson random variable with parameter 6. There are 4 parking spaces on the street.*

   (a) What is the probability that the number of cars parking on that street is exactly 3?

   This number is $e^{-6}\frac{6^3}{3!}$. [=0.0892.]

   (b) What is the probability that all the parking spots are taken?

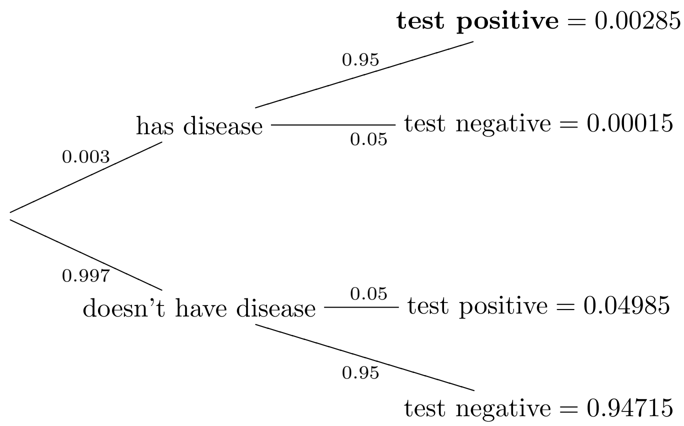
   Let X be the number of cars that want to park in the street. This is $P(X \geqslant 4) = 1 - P(X = 0) - P(X = 1) - P(X = 2) - P(X = 3) = 1 - e^{-6}\left(1 + 6 + \frac{6^2}{2!} + 6^3 3!\right) = 1 - e^{-6} \times 61 = 1 - 0.151 = 0.849$.

   (c) What is the expected number of free parking spaces?

Number of free spaces	Probability	Number of free spaces times probability
4	e^{-6}	$4e^{-6}$
3	$6e^{-6}$	$18e^{-6}$
2	$18e^{-6}$	$36e^{-6}$
1	$36e^{-6}$	$36e^{-6}$
0	$1 - 61e^{-6}$	0
Total	1	$94e^{-6}$

So the expected number of free parking spaces is $94e^{-6} = 0.233$.

9. *A patient is given a routine test for a rare disease. The disease affects 3 people in 1000. The test is 95% accurate, so there is a 5% chance of giving the wrong result. The test result is positive (i.e. indicates the patient has the disease). What is the probability that the patient actually has the disease?*



So the total probability of a positive test result is $0.04985 + 0.00285 = 0.0527$, and the probability of having the disease given a positive test result is $\frac{0.00285}{0.0527} = 0.0541$.

10. *A company is conducting a survey. They want to determine the proportion of people who would buy their new product. If the true proportion is 30%, how many people do they need to survey so that the probability that their estimate is within 2% of the true value (i.e. between 28% and 32%) is at least 95%?*

If the survey includes n people, then the expected number of answers is $0.3n$, and the variance is $0.3 \times 0.7n = 0.21n$. We can approximate the distribution as a normal distribution. The probability of a standard normal being within x of 0 is 95% when $\Phi(x) = 0.975$, which happens when $x = 1.96$. Therefore, we want $1.96\sqrt{0.21n} = 0.02n$. This gives $0.806736n = 0.0004n^2$, or $n = 2016.84$, so they need to survey at least 2017 people.

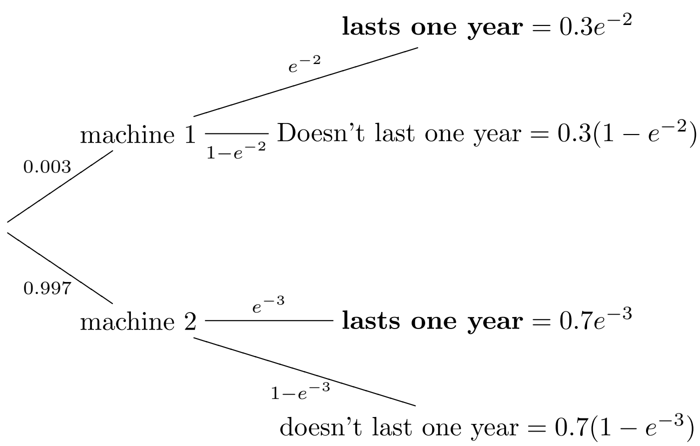
11. *A printer contains a pair of rollers, between which a sheet of paper passes. One roller has diameter 5cm and the other has diameter 7cm. Each roller has a defect that affects 1cm of the circumference. When the paper is between both defects, it rips. A sheet of paper is 28cm long. What is the probability that a random sheet of paper gets ripped? [Hint: the positions of the rollers repeat every 5 rotations of the larger roller.]*

The total distance in a complete cycle of the two rollers is 5 cycles of the larger roller, or 35πcm. The paper enters at one uniformly distributed point in this cycle. If the point in the cycle where the paper enters is in the 1cm defective region or in the 28cm before it, then the paper will rip. This happens with probability $\frac{29}{35\pi}$.[= 0.264]

12. *A company makes light bulbs. The company has two machines for making them. light bulbs made by one machine have lifetime (in years) exponentially distributed with parameter 2, and light bulbs made by the other machine have lifetime exponentially distributed with parameter 3. 30% of its products are made by the first machine.*

    (a) What is the probability that a randomly chosen light bulb lasts for at least 1 year?

    If the light bulb is produced by machine 1, then the probability is e^{-2}, while if the light bulb is produced by machine 2, then the probability is e^{-3}.



    So the total probability of lasting one year is $0.3e^{-2} + 0.7e^{-3}$. [= 0.0754]

    (b) Given that it lasts for 1 year, what is the probability that it was produced by machine 1?

    The probability that a light bulb was produced by machine 1 and lasts one year is $0.3e^{-2}$. The total probability that it lasts one year is $0.3e^{-2} + 0.7e^{-3}$. Therefore, the probability that it was from machine 1 given that it lasts a year is $\frac{0.3e^{-2}}{0.3e^{-2}+0.7e^{-3}} = 0.538$.

    (c) Given that a light bulb has lasted for 1 year, what is the probability that it lasts for a second year?

    The same argument as in (a) gives that this probability is $0.538e^{-2} + 0.462e^{-3} = 0.0958$.

13. *A company produces 50,000,000 scratchcards, which it will sell for \$1 each. The scratchcards offer the following cash prizes:*

Prize	*Number of cards with this prize*
0	*48,000,000*
\$10	*1,850,000*
\$100	*140,000*
\$1,000	*9,750*
\$10,000	*235*
\$100,000	*14*
\$1,000,000	*1*

(a) What is the expected value and variance for the prize of a scratch card?

Prize	Probability	Prize × Probability	Prize^2 × Probability
0	0.96	0	0
\$10	0.037	0.37	3.7
\$100	0.0028	0.28	28
\$1,000	0.000195	0.195	195
\$10,000	0.0000047	0.047	470
\$100,000	0.00000028	0.028	2,800
\$1,000,000	0.00000002	0.02	20,000
total		0.94	23,496.7

The expected value is therefore \$0.94, and the variance is $23,496.7 - 0.94^2 = 23,495.8836$.

(b) If the company sells 100,000 scratch cards, what is the expected profit?

The income from the scratchcards sold is \$100,000. The expected total of prizes paid out is $100,000 \times 0.94 = \$94,000$, so the expected profit is \$6,000.