MATH/STAT 3360, Probability
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Model Solutions

1. *X is normally distributed with mean 2 and standard deviation 3. Y is normally distributed with mean 4 and standard deviation 4. What is $P(Y > X)$?*

   (a) If X and Y are independent?

   If X and Y are independent, then $Y - X$ is normally distributed with mean $4 - 2 = 2$ and variance $3^2 + 4^2 = 25$, so it has standard deviation 5. We want $P(Y - X > 0)$. Let $Z = \frac{Y-X-2}{5}$, then $Z \sim N(0,1)$, and we are looking for $P(Z > -0.4) = 1 - \Phi(-0.4) = \Phi(0.4) = 0.6554$.

   (b) If X, Y have a multivariate normal distribution, and $\mathrm{Cov}(X,Y) = 2$.

   Again we have $Y - X$ is normally distributed with mean 2, but now the variance is $3^2 + 4^2 - 2\,\mathrm{Cov}(X,Y) = 21$, so $P(Y - X > 0) = \Phi\left(\frac{2}{\sqrt{21}}\right) = \phi(0.44) = 0.6700$.

2. *(R, Θ) have joint density function $f_{(R,\Theta)}(r,\theta) = \frac{1}{2\pi}e^{-r}$. Let $X = R\cos\Theta$ and $Y = R\sin\Theta$.*

   (a) What is the joint density function of (X,Y)?

   The Jacobian matrix is

   $$\begin{pmatrix} -r\sin\theta & \cos\theta \\ r\cos\theta & \sin\theta \end{pmatrix}$$

   so its determinant is $-r\sin^2\theta - r\cos^2\theta = -r$.

   We therefore have $f_{X,Y}(r\cos\theta, r\sin\theta) = \frac{f_{R,\Theta}(r,\theta)}{r}$ (where r is always positive, so there is no need to take its absolute value).

   This gives $f_{X,Y}(r\cos\theta, r\sin\theta) = \frac{e^{-r}}{2\pi r}$. We now write $r = \sqrt{x^2+y^2}$ to get

   $$f_{X,Y}(x,y) = \frac{e^{-\sqrt{x^2+y^2}}}{2\pi\sqrt{x^2+y^2}}$$

   (b) Are X and Y independent?

   X and Y are not independent, because this probability density function cannot be written as a product of a function of X and a function of Y.

3. *X is normally distributed with mean 0 and variance 1. Y is independent of X and has probability density functions*

   $$f_Y(y) = \frac{\lambda}{2}e^{-\lambda|y|}$$

for some constant $\lambda > 0$. Find the joint density function of $Z = X + Y$ and $W = X + 3Y$.

The joint density function of X and Y is

$$f_{X,Y}(x, y) = \frac{\lambda}{2\sqrt{2\pi}} e^{-\frac{x^2}{2}} e^{-\lambda|y|}$$

We have that the Jacobian matrix for the function sending (x, y) to (z, w) is

$$\begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}$$

so its determinant is $1 \times 3 - 1 \times 1 = 2$. We therefore have $f_{Z,W}(z, w) = \frac{f_{X,Y}(x,y)}{2}$, furthermore, by solving the equations we have $x = \frac{3z-w}{2}$ and $y = \frac{w-z}{2}$, so we get

$$f_{Z,W}(z, w) = \frac{\lambda}{4\sqrt{2\pi}} e^{-\frac{(3z-w)^2}{8} - \frac{\lambda|w-z|}{2}}$$

4. *Let X have a Poisson distribution with parameter λ, and given that $X = n$, let Y have a binomial distribution with parameters n and p.*

(a) What is the joint probability mass function of X and Y

We have $P(X = n, Y = i) = e^{-\lambda} \frac{\lambda^n}{n!} \binom{n}{i} p^i (1-p)^{n-i}$, whenever $i \leqslant n$.

[Alternatively, we can write this as $P(X = n, Y = i) = e^{-\lambda} \frac{\lambda^n}{i!(n-i)!} p^i (1 - p)^{n-i} = e^{-\lambda} \frac{(\lambda p)^i}{i!} \frac{(\lambda(1-p))^{n-i}}{(n-i)!}$.]

(b) Show that the marginal distribution of Y is a Poisson distribution with parameter λp.

The marginal distribution of Y is given by

$$P(Y = i) = \sum_{n=i}^{\infty} \left(e^{-\lambda} \frac{(\lambda p)^i}{i!} \frac{(\lambda(1-p))^{n-i}}{(n-i)!} \right)$$

We substitute $m = n - i$ in the sum to get

$$P(Y = i) = \sum_{m=0}^{\infty} \left(e^{-\lambda} \frac{(\lambda p)^i}{i!} \frac{(\lambda(1-p))^m}{m!} \right) = e^{-\lambda p} \frac{(\lambda p)^i}{i!} e^{-\lambda(1-p)} \sum_{m=0}^{\infty} \frac{((\lambda(1-p))^m)}{m!} = e^{-\lambda p} \frac{(\lambda p)^i}{i!}$$

(c) What is the conditional distribution of X conditional on $Y = i$?

We have $P(X = n|Y = i) = \frac{P(X=n,Y=i)}{P(Y=i)} = \frac{\left(e^{-\lambda} \frac{(\lambda p)^i}{i!} \frac{(\lambda(1-p))^{n-i}}{(n-i)!} \right)}{e^{-\lambda p} \frac{(\lambda p)^i}{i!}} = e^{-\lambda(1-p)} \frac{(\lambda(1-p))^{n-i}}{(n-i)!}$

[So the conditional distribution of X is i plus a Poisson random variable with parameter $\lambda(1-p)$.]

5. *Let X and Y be independent uniform random variables on intervals $[2,4]$ and $[3,4]$ respectively. Calculate the probability density function of $X+Y$.*

   We calculate this by convolution:

$$f_{X+Y}(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)$$

   This integrand $f_x(x)f_Y(z-x)$ is zero unless:

   - $5 \leqslant z \leqslant 6$, and $2 \leqslant x \leqslant z-3$.
   - $6 \leqslant z \leqslant 7$, and $z-4 \leqslant x \leqslant z-3$.
   - $7 \leqslant z \leqslant 8$, and $z-4 \leqslant x \leqslant 4$.

   In these cases, we have $f_X(x) = 0.5$ and $f_Y(y) = 1$, so that

$$f_{X+Y}(z) = \begin{cases} 0 & \text{if } z < 5 \\ \frac{z-3}{2} & \text{if } 5 \leqslant z \leqslant 6 \\ \frac{1}{2} & \text{if } 6 \leqslant z \leqslant 7 \\ \frac{8-z}{2} & \text{if } 7 \leqslant z \leqslant 8 \\ 0 & \text{if } z \geqslant 8 \end{cases}$$

6. *The point (X,Y) is uniformly distributed on the set $\{(x,y)|x+y<3, y> -1, x>-2\}$.*

   (a) What is the conditional expectation of X given that $Y=2$?

   Given that $Y=2$, we have $x+2<3$ and $x>-2$, so we get $-2<x<1$, and the probability density function is constant on this set, so X is uniformly distributed on $[-2,1]$. Its conditional expectation is therefore $\frac{1-2}{2} = -\frac{1}{2}$.

   (b) What is $\mathrm{Cov}(X,Y)$?

   Let A be the set $\{(x,y)|x+y<3, y>-1, x>-2\}$. Then A is a triangle with vertices at $(-2,5)$, $(-2,-1)$ and $(4,-1)$. The area of A is therefore, 18, so $f_{X,Y}(x,y) = \frac{1}{18}$ whenever $(x,y) \in A$.

   We have that $\mathbb{E}(X) = \int\int_A \frac{x}{18} dxdy = \int_{-1}^{5}\int_{-2}^{3-y} \frac{x}{18} dxdy = \int_{-1}^{5} [\frac{x^2}{36}]_{-2}^{3-y} dy = \int_{-1}^{5} \frac{(3-y)^2-4}{36} dy = \left[\frac{(y-3)^3}{108}\right]_{-1}^{5} - \frac{2}{3} = \frac{8-(-64)}{108} - \frac{2}{3} = -\frac{2}{108} = -\frac{1}{54}$. Similarly, we get $\mathbb{E}(Y) = \frac{53}{54}$. Now $\mathbb{E}(XY) = \int\int_A \frac{xy}{18} dxdy = \int_{-1}^{5} \frac{y((y-3)^2-4)}{36} dy = \int_{-1}^{5} \frac{y^3-6y^2+5y}{36} dy = \left[\frac{y^4-8y^3+10y^2}{144}\right]_{-1}^{5} = \frac{625-1000+250-1-8-10}{144} = -1$. We then get $\mathrm{Cov}(X,Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) = -1 + \frac{53}{54^2} = -\frac{2863}{2916}$.

7. *If X is exponentially distributed with parameter λ and Y is independent of X and normally distributed with mean μ and variance σ^2, where $\mu < 0$.*

(a) find the moment generating function of $X - Y$.

The moment generating function of X is $M_X(t) = \frac{\lambda}{\lambda+t}$. The moment generating function of Y is $M_Y(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$, so the moment generating function of $X - Y$ is $M_{X-Y}(t) = M_X(t)M_Y(-t) = \frac{\lambda e^{\frac{\sigma^2 t^2}{2} - \mu t}}{\lambda - t}$.

(b) Use the Chernoff bound with $t = \frac{2\mu}{\sigma^2}$ to obtain a lower *bound on the probability that $X > Y$.*

The Chernoff bound for $t < 0$ gives $P(X - Y \leqslant 0) \leqslant e^{-0t} M_{X-Y}(t)$ Setting $t = \frac{2\mu}{\sigma^2}$, then this gives, $P(X - Y \leqslant 0) \leqslant \frac{\lambda e^{\frac{4\mu^2}{2\sigma^2} - \frac{2\mu^2}{\sigma^2}}}{\lambda - \frac{2\mu}{\sigma^2}} = \frac{\lambda}{\lambda - \frac{2\mu}{\sigma^2}}$. Now $P(X > Y) = 1 - P(X - Y \leqslant 0) \geqslant 1 - \frac{\lambda}{\lambda - \frac{2\mu}{\sigma^2}} = \frac{-2\mu}{\sigma^2\lambda - 2\mu}$

8. *Let X have an exponential distribution with parameter λ. Suppose that given $X = x$, Y is normally distributed with mean x and variance σ^2.*

(a) What is the joint density function of X and Y?

We have that $f_{X,Y}(x, y) = f_X(x) f_{Y|X}(y) = \frac{\lambda}{\sqrt{2\pi}\sigma} e^{-\lambda x} e^{-\frac{(y-x)^2}{2\sigma^2}}$.

(b) What is the covariance of X and Y?

We know that $\mathbb{E}(X) = \frac{1}{\lambda}$. On the other hand, we have that $\mathbb{E}(Y|X = x) = x$, so $\mathbb{E}(Y) = \mathbb{E}(\mathbb{E}(Y|X)) = \mathbb{E}(X) = \frac{1}{\lambda}$.

Finally, we consider $\mathbb{E}(XY|X = x) = x\mathbb{E}(Y|X = x) = x^2$, so $\mathbb{E}(XY) = \mathbb{E}(X^2) = \frac{2}{\lambda^2}$. Therefore, we have $\mathrm{Cov}(X, Y) = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$.

(c) Conditional on $Y = y$, what is the density function of X? [You do not need to calculate the constant factor.]

Conditional on $Y = 3$, we get $f_{X|Y}(x|y = 3) = Ce^{-\lambda x} e^{-\frac{(x-y)^2}{2\sigma^2}} = Ce^{-\frac{x^2 + 2(y - \lambda\sigma^2)x + y^2)}{2\sigma^2}}$, provided $x > 0$ and $f_{X|Y}(x|y = 3) = 0$ for $x < 0$. [So X has a truncated normal distribution with mean $y - \lambda\sigma^2$, and variance σ^2.]

9. *Consider the following experiment: Toss 14 fair coins. For each toss that results in a head, roll a fair (6-sided) die. Take the sum of the numbers rolled on all these dice. What is the expected outcome?*

Let H be the number of heads. Let X be the outcome. We know that $H \sim B(14, \frac{1}{2})$, and that $\mathbb{E}(X|H = n) = 3.5n$. We therefore have $\mathbb{E}(X) = \mathbb{E}(\mathbb{E}(X|H)) = \mathbb{E}(3.5H) = 3.5\mathbb{E}(H) = 3.5 \times 0.5 \times 14 = 24.5$.

10. *In a class of 100 students, the professor wants to determine what proportion of the students can answer a simple calculus question. The professor decides to test a random sample of 10 students. In fact 35 of the students could answer the question.*

(a) What is the expected number of students sampled who answer the question correctly?

Let the students who know the answer be numbers 1 to 35. Let X_i be the number of times each student is chosen. We have that $\mathbb{E}(X_i) = \frac{10}{100} = 0.1$, for each i. The number of students who answer correctly is $\sum_{i=1}^{35} X_i$, so the expected number is $\sum_{i=1}^{35} \mathbb{E}(X_i) = 35 \times 0.1 = 3.5$.

(b) What is the variance of the number of correct answers if the students are sampled with replacement, i.e. one student could be tested more than once?

Let $Y_1, \ldots, Y_{10}$ be indicator random variables for the events that the students tested know the answer. If the students are sampled with replacement, then we have that each Y_i is 1, and the Y_i are independent. We therefore have $\mathrm{Var}(Y_1 + \cdots + Y_{10}) = \mathrm{Var}(Y_1) + \cdots + \mathrm{Var}(Y_{10})$. Each Y_i is 1 with probability 0.35 and zero with probability 0.65, so $\mathbb{E}(Y_i) = 0.35$ and $\mathbb{E}({Y_i}^2) = 0.35$, so $\mathrm{Var}(Y_i) = 0.35 \times 0.65 = 0.2275$. Therefore, the variance of the number of correct answers is 2.275.

(c) What is the variance of the number of correct answers if the 10 students sampled must be 10 different students?

If the students must be different, then each X_i is either zero or one. Furthermore, for $i \neq j$, the probability that students i and j are both selected is $\frac{\binom{98}{8}}{\binom{100}{10}} = \frac{1}{110}$. We now have $\mathbb{E}((\sum_{i=1}^{35} X_i)^2) = \sum_{i=1}^{35} \sum_{j=1}^{35} \mathbb{E}(X_i X_j) = 35 \times 0.1 + \frac{34 \times 35}{110} = \frac{157.5}{11}$. We therefore have that the variance of the number of correct answers is $\frac{157.5}{11} - 3.5^2 = \frac{157.5 - 134.75}{11} = \frac{22.75}{11} = 2.068$.

11. *An ecologist is collecting snails. She collects a total of 40 snails. There are a total of 18 species of snail, and each snail is equally likely to be any of the species.*

(a) What is the expected number of species of snails she collects?

Let $X_1, \ldots, X_{18}$ be indicator variables, to indicate that she did not collect any snails of species i. That is:

$$X_i = \begin{cases} 1 & \text{if she does not collect any snails of species } i \\ 0 & \text{if she collects at least one snail of species } i \end{cases}$$

Now we have $\mathbb{E}(X_i) = P(X_i) = \left(\frac{17}{18}\right)^{40}$, so the total number of species of snails she collects is $18 - X = 18 - (X_1 + \cdots + X_{18})$, and the $\mathbb{E}(X) = 18\left(1 - \left(\frac{17}{18}\right)^{40}\right) = 16.17$.

(b) What is the variance of the number of species of snails she collects?

The variance of the number of species she collects is the same as the variance of the number of species she doesn't collect. We have $\mathbb{E}(X^2) = \sum_{i=1}^{18} \sum_{j=1}^{18} \mathbb{E}(X_i X_j)$. If $i = j$, we know that $\mathbb{E}(X_i X_j) = \mathbb{E}(X_i) = \left(\frac{17}{18}\right)^{40}$, while if $i \neq j$, we have $\mathbb{E}(X_i X_j) = \left(\frac{16}{18}\right)^{40}$, so that $\mathbb{E}(X^2) = 18 \times 17 \times \left(\frac{16}{18}\right)^{40} + 18 \times \left(\frac{17}{18}\right)^{40}$ and $\mathrm{Var}(X) = 18 \times 17 \times \left(\frac{16}{18}\right)^{40} + 18 \times \left(\frac{17}{18}\right)^{40} - 18^2 \times \left(\frac{17}{18}\right)^{80} = 1.234$.

(c) Using the one-sided Chebyshev inequality, find a bound on the probability that she collects all 18 species. [Hint: The Chebyshev inequality will give the probability of collecting at least 18 species.]

The one-sided Chebyshev inequality states that for $a > 0$, $P(X - \mathbb{E}(X) \geqslant a) \leqslant \frac{\mathrm{Var}(X)}{\mathrm{Var}(X)+a^2}$. In this case, it tells us that $P((18 - X) - \mathbb{E}(18 - X) \geqslant 18 - \mathbb{E}(18 - X)) \leqslant \frac{\mathrm{Var}(X)}{\mathrm{Var}(X)+(\mathbb{E}(X))^2} = 0.269$. So the probability that she collects all the species is at most 0.269.

12. *A car insurance company finds that of its claims, 70% are for accidents, and 30% are for thefts. The theft claims are all for $20,000, while for the accident claims claim, the expected amount claimed is $15,000, and the standard deviation of the amount claimed is $30,000.*

    (a) What are the expected amount claimed, and the standard deviation of the amount claimed for any claim?

    Let X be the amount claimed. Let $Y = 1$ if the claim is for an accident, and 0 for a theft. We have $\mathbb{E}(X|Y = 1) = 15000$ and $\mathbb{E}(X|Y = 0) = 20000$, so $\mathbb{E}(X) = \mathbb{E}(\mathbb{E}(X|Y)) = 0.7 \times 15000 + 0.3 \times 20000 = 16500$.

    For the variance, we have $\mathrm{Var}(X) = \mathbb{E}(\mathrm{Var}(X|Y)) + \mathrm{Var}(\mathbb{E}(X|Y)) = 0.7 \times 30000^2 + 0.7 \times 15000^2 + 0.3 \times 20000^2 - 16500^2 = 635250000$. The standard deviation is therefore $\sqrt{635250000} = 25204.166$.

    (b) If the company has $16,700,000 available to cover claims, and receives 1000 claims, what is the probability that it is unable to cover the claims made?

    This is the probability that the average amount claimed X is more than $16,700. By the central limit theorem, the average amount claimed can be approximated as a normal distribution with mean $16,500 and standard deviation $\frac{25204.166}{\sqrt{1000}} = 797.03$. We let $Z = \frac{X-16500}{797.03}$, so $P(X > 16700) = P\left(Z > \frac{16700-16500}{797.03}\right) = 1 - \Phi(0.25) = 1 - 0.5987 = 0.4013$.

13. *You are considering an investment. You would originally invest $1,000, and every year, the investment will either increase by 50% with probability 0.6 or decrease by 30% with probability 0.4. You plan to use the investment after 15 years. What is the expected value of the investment after 15 years?*

    Let X_i be the value of the investment after i years. We know that $X_0 = 1000$ and conditional on $X_i = x$, we have

$$X_{i+1} = \begin{cases} 1.5X & \text{with probability } 0.6 \\ 0.7X & \text{with probability } 0.4 \end{cases}$$

This gives $\mathbb{E}(X_{i+1}|X_i = x) = 0.6 \times 1.5 \times x + 0.4 \times 0.7x = 1.18x$. We therefore have $\mathbb{E}(X_{i+1}) = \mathbb{E}(\mathbb{E}(X_{i+1}|X_i)) = \mathbb{E}(1.18X_i) = 1.18\mathbb{E}(X_i)$. From this we deduce that $\mathbb{E}(X_i) = 1000(1.18)^i$, so $\mathbb{E}(X_{15}) = 11973.75$.

14. *A company is planning to run an advertising campaign. It estimates that the number of customers it gains from the advertising campain will be approximately normally distributed with mean 3,000 and standard deviation 300. It also estimates that the amount spent by each customer has expected value \$50 and standard deviation \$30.*

    (a) Assuming the number of customers is large enough, what is the approximate distribution of the average amount spent per customer, conditional on the number of new customers being n.

    The average amount spent per customer is normally distributed with mean \$50 and standard deviation $\frac{30}{\sqrt{n}}$.

    (b) What is the joint density function for the number of new customers and the average amount spent per customer?

    The joint density function is given by

$$f_{X,Y}(x,y) = f_X(x)f_{Y|X}(y|X=x) = \frac{1}{\sqrt{2\pi}300}e^{-\frac{(x-3000)^2}{2\times 300^2}}\frac{1}{\sqrt{2\pi}\frac{30}{\sqrt{x}}}e^{-\frac{(y-50)}{\left(\frac{2\times 30^2}{x}\right)}}$$

$$= \frac{\sqrt{x}}{18000\pi}e^{-\frac{(x-3000)^2}{180000}-\frac{x(y-50)^2}{1800}}$$

15. *A gambler is playing a slot machine in a casino. The gambler continues to bet \$1 each time. The machine has the following payouts:*

Payout	*Probability*
\$1	*0.4*
\$10	*0.03*
\$100	*0.002*
\$1000	*0.00005*

    How many times does the gambler have to play before the probability of his having more money than he started with is less than 1%?

    Let X_i be the amount the gambler gains on the ith play (the amount he wins minus his original \$1). If W_i is the amount he wins, then we have $\mathbb{E}(X_i) = \mathbb{E}(W_i) - 1$ and $\mathrm{Var}(X_i) = \mathrm{Var}(W_i)$.

    Now $\mathbb{E}(W_i) = 0.4 \times 1 + 0.03 \times 10 + 0.002 \times 100 + 0.00005 \times 1000 = 0.95$ and $\mathbb{E}({X_i}^2) = 0.4 \times 1 + 0.03 \times 100 + 0.002 \times 10000 + 0.00005 \times 1000000 = 73.4$, so $\mathrm{Var}(W_i) = 73.4 - 0.95^2 = 72.4975$. If the gambler plays n times, for large enough n, the distribution of his average gain per play A_n, is normal with mean -0.05 and variance $\frac{72.4975}{n}$. The probability that he has more money than he started with is therefore $P(A_n > 0) = 1 - \Phi\left(\frac{0.05}{\sqrt{\frac{72.4975}{n}}}\right)$. From the normal table, we have $\Phi(2.33) = 0.99$, so we need to solve

$$0.05\sqrt{\frac{n}{72.4975}} = 2.33$$

which gives $n = 72.4975 \times \frac{2.33^2}{0.05^2} = 157433$.