MATH/STAT 3360, Probability
FALL 2013
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Sample Midterm Examination
Model Solutions

This Sample Midterm has more questions than the actual midterm, in order to cover a wider range of questions.

1. *How many distinct ways can the letters of the word " EXAMINATION" be arranged?*

   There are 11 letters, of which 2 are 'A', 2 are 'I', and 2 are 'N', so the number of ways they can be arranged is $\frac{11!}{2!2!2!} = 4989600$.

2. *What is the probability that the sum of 3 fair 6-sided dice is 6?*

   The possibilities where the sum is 6 are:

   114, 123, 132, 141, 213, 222, 231, 312, 321, and 411. The probability is therefore $\frac{10}{216} = 0.0463$.

3. *Calculate the probability density function of the square of an exponential random variable with parameter λ.*

   If $X \sim \text{Exp}(\lambda)$, then the probability density function of X^2 is given by $f_{X^2}(x) = \frac{\lambda e^{-\lambda\sqrt{x}}}{2\sqrt{x}}$.

4. *What is the probability that a five-card poker hand is a flush (all cards the same suit)?*

   There are $4 \times \binom{13}{5}$ hands which are flushes (4 suits and $\binom{13}{5}$ flushes for each). Therefore, the probability of a flush is $\frac{4\times\binom{13}{5}}{\binom{52}{5}} = 0.00198$.

5. *A fair coin is tossed 7 times. What is the probability that the sequence HHTT occurs somewhere in the 7 tosses?*

   There are 4 possible positions where the sequence could occur. Let E_1, E_2, E_3, E_4 be the events that the sequence occurs starting in the first, second, third and fourth position respectively. These events are mutually exclusive, so $P(E_1 \cup E_2 \cup E_3 \cup E_4) = P(E_1) + P(E_2) + P(E_3) + P(E_4) = \frac{4}{16} = \frac{1}{4}$.

6. *Three coins are tossed: are the following events independent?*

   (i) The first two tosses are both heads.

   (ii) The total number of heads is odd.

   Let A be the event that The first two tosses are both heads, and let B be the event that The total number of heads is odd. Clearly we have that $P(A) = \frac{1}{4}$, $P(B) = \frac{1}{2}$, and $P(A \cap B) = \frac{1}{8}$, so the events are independent.

Alternatively, for any outcome of the first three tosses, the conditional probability of B is $\frac{1}{2}$, so $P(B|A) = \frac{1}{2} = P(B)$, so A and B are independent.

7. *Suppose the number of cars that want to park in a particular street each day is a Poisson random variable with parameter 4. There are 3 parking spaces on the street.*

   (a) What is the probability that the number of cars parking on that street is exactly 2?

   Let N be the number of cars. $P(N = 2) = e^{-4}\frac{4^2}{2!} = 0.1465$.

   (b) What is the probability that all the parking spots are taken?

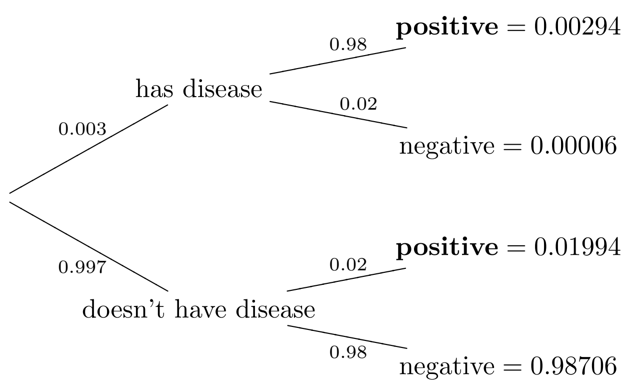
   All the parking spots are taken if $N \geqslant 3$, so $P(N \geqslant 3) = 1 - P(N \leqslant 2) = 1 - P(N = 0) - P(N = 1) - P(N = 2) = 1 - e^{-4}(1 + 4 + 8) = 0.762$.

   (c) What is the expected number of free parking spaces?

   let F be the number of free parking spaces: We have that $P(F = 3) = e^{-4}$, $P(F = 2) = 4e^{-4}$ and $P(F = 1) = \frac{4^2}{2}e^{-4}$, so $\mathbb{E}(F) = 3e^{-4} + 2 \times 4e^{-4} + 1 \times 8e^{-4} = 19e^{-4} = 0.348$.

8. *A patient is given a routine test for a rare disease. The disease affects 3 people in 1000. The test is 98% accurate, so there is a 2% chance of giving the wrong result. The test result is positive (i.e. indicates the patient has the disease). What is the probability that the patient actually has the disease?*

   In a tree diagram:



   So the probability that the patient actually has the disease is $\frac{0.00294}{0.00294+0.01994} = \frac{1}{20.99998} = 0.128$.

9. *A company is conducting a survey. They want to determine the proportion of people who would buy their new product. The true proportion is 30%.*

   [You may use any reasonable approximations for the distribution of the proportion of people surveyed who say they would buy it.]

(a) If they survey 300 people, what proportion has a 95% probability of being larger than their estimate? That is, find x so that there is a 95% probability that their estimate is less than x.

The number of people N who say they would buy the product has a binomial distribution with mean $300 \times 0.3 = 90$ and variance $300 \times 0.3 \times 0.7 = 63$. We approximate this distribution as a normal distribution with mean 90 and variance 63. We then define $Z = \frac{N-90}{\sqrt{63}}$, so that $Z \sim N(0,1)$. We know from the normal table that $\Phi(1.645) = 0.95$, so we have $P(Z < 1.645) = 0.95$, and so we get $P(N < 1.645 \times \sqrt{63} + 90 = 103) = 0.95$. Therefore, the proportion is $\frac{103}{300} = 34.35\%$.

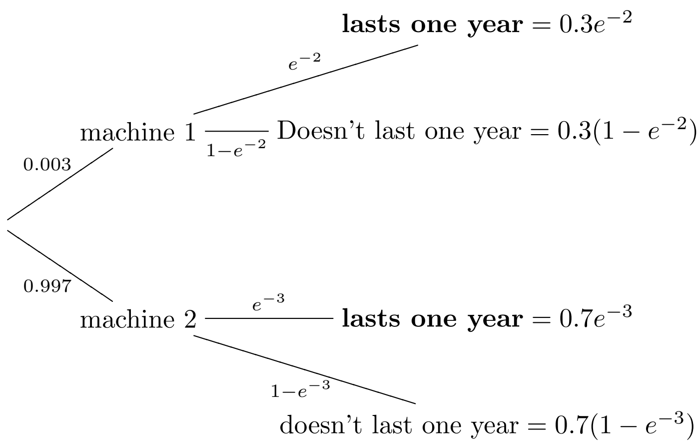
(b) how many people do they need to survey so that the probability that their estimate is within 2% of the true value (i.e. between 28% and 32%) is at least 95%?

If they survey n people, N has a binomial distribution with parameters n and 0.3. We approximate this as $N(0.3n, 0.21n)$, and the proportion they find is $\frac{N}{n}$, which is approximately distributed $N\left(0.3, \frac{\sqrt{0.21}}{\sqrt{n}}\right)$. We want to find n so that the probability that this is between 0.28 and 0.32 is at least 95%. We let $Z = \frac{\frac{N}{n}-0.3}{\frac{\sqrt{0.21}}{\sqrt{n}}}$. Now $Z \sim N(0,1)$, so we find z so that $\Phi(z) - \Phi(-z) = 0.95$. This gives $\Phi(z) = 0.975$, and from the table, we see $z = 1.96$. Therefore, $P(-1.96 < Z < 1.96) = 0.95$. We therefore want $\frac{0.32-0.3}{\frac{\sqrt{0.21}}{\sqrt{n}}} = 1.96$. We solve this to get $\sqrt{n} = \frac{1.96\sqrt{0.21}}{0.02}$, so $n = 2016.84$. is the number of people they need to survey.

10. *A company makes light bulbs. The company has two machines for making them. light bulbs made by one machine have lifetime (in years) exponentially distributed with parameter 2, and light bulbs made by the other machine have lifetime exponentially distributed with parameter 3. 30% of its products are made by the first machine.*

    (a) What is the probability that a randomly chosen light bulb lasts for at least 1 year?

    If the light bulb is produced by machine 1, then the probability is e^{-2}, while if the light bulb is produced by machine 2, then the probability is e^{-3}.



So the total probability of lasting one year is $0.3e^{-2} + 0.7e^{-3}$. [$= 0.0754$]

(b) Given that it lasts for 1 year, what is the probability that it was produced by machine 1?

The probability that a light bulb was produced by machine 1 and lasts one year is $0.3e^{-2}$. The total probability that it lasts one year is $0.3e^{-2} + 0.7e^{-3}$. Therefore, the probability that it was from machine 1 given that it lasts a year is $\frac{0.3e^{-2}}{0.3e^{-2}+0.7e^{-3}} = 0.538$.

(c) Given that a light bulb has lasted for 1 year, what is the probability that it lasts for a second year?

The same argument as in (a) gives that this probability is $0.538e^{-2} + 0.462e^{-3} = 0.0958$.

11. *An insurance company sells 12,000 policies, each of which has probability $\frac{1}{8{,}000}$ of resulting in a claim. What is the probability that it receives more than 2 claims?*

    Let N be the number of claims it receives. N has a binomial distribution with parameters 12,000 and $\frac{1}{8000}$. Therefore $P(N > 2) = 1 - P(N = 0) - P(N = 1) - P(N = 2) = 1 - \left(\frac{7999}{8000}\right)^{12000} - 12000\left(\frac{7999}{8000}\right)^{11999}\left(\frac{1}{8000}\right) - \binom{12000}{2}\left(\frac{7999}{8000}\right)^{11998}\left(\frac{1}{8000}\right)^2 = 0.191$.

12. *A company produces 50,000,000 scratchcards, which it will sell for \$1 each. The scratchcards offer the following cash prizes:*

Prize	*Number of cards with this prize*
0	*48,000,000*
\$10	*1,850,000*
\$100	*140,000*
\$1,000	*9,750*
\$10,000	*235*
\$100,000	*14*
\$1,000,000	*1*

What is the expected value and variance for the prize of a scratch card?

Let X be the prize on a scratchcard. We have

x	$P(X = x)$	$xP(X = x)$	$x^2P(X = x)$
10	0.037	0.37	3.7
100	0.0028	0.28	28.0
1,000	0.000195	0.195	195.0
10,000	0.0000047	0.047	470.0
100,000	0.00000028	0.028	2,800.0
1,000,000	0.00000002	0.02	20,000.0
total		0.94	23496.7

Therefore $\mathbb{E}(X) = 0.94$ and $\mathrm{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = 23496.7 - 0.94^2 = 23495.8164$.