MATH/STAT 3360, Probability
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Toby Kenney
Midterm Examination
Model Solutions

1. *What is the probability that a five-card poker hand is a two-pair (two cards of one rank, two of another, and one of a third rank)?*

   The number of such hands is $\binom{13}{2}\binom{4}{2}^2 44$, so the probability is $\frac{44\binom{13}{2}\binom{4}{2}^2}{\binom{52}{5}}$.

2. *A fair coin is tossed 8 times. What is the probability that the sequence THTT occurs somewhere in the 8 tosses?*

   There are 5 possible positions where the sequence could occur. Let the corresponding events be E_1, E_2, E_3, E_4 and E_5. It is easy to see that any pair of these are mutually exclusive, with the exceptions of E_1 and E_4; E_1 and E_5 and E_2 and E_5. We see that $P(E_i) = \frac{1}{16}$, $P(E_1 \cap E_4) = \frac{1}{128}$, $P(E_1 \cap E_5) = \frac{1}{256}$, and $P(E_2 \cap E_5) = \frac{1}{128}$. Therefore, $P(E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5) = \frac{5}{16} - \frac{1}{128} - \frac{1}{256} - \frac{1}{128} = \frac{75}{256}$.

3. *Four fair coins are tossed. Are the following events independent:*

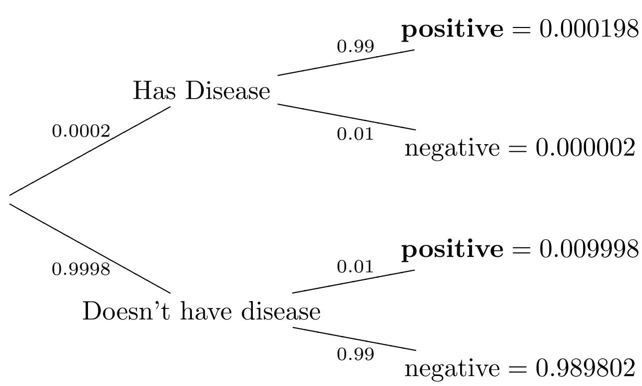
   (i) The first toss is a head

   (ii) There are exactly two heads.

   Let A be the event that the first toss is a head, and B be the event that there are exactly two heads. We have $P(A) = \frac{1}{2}$, $P(B) = \frac{6}{16}$, and $P(A \cap B) = \frac{3}{16} = P(A)P(B)$, so the events are independent.

4. *A patient is given a routine test for a rare disease. The disease affects 1 person in 5000. The test is 99% accurate, so there is only a 1% chance of giving the wrong result. The test result is positive (i.e. indicates the patient has the disease). What is the probability that the patient actually has the disease?*

   In a tree diagram:



So the probability that the patient actually has the disease is $\frac{0.000198}{0.000198+0.009998} = 0.0194$.

5. *A company is conducting a survey. They want to determine the proportion of people who would buy their new product.*

   (a) If the true proportion is 40%, and they survey 300 people, what is the probability that their estimate is within 1% of the true value (i.e. between 39% and 41%)?

   [You may use any reasonable approximation for the distribution of the number of people who say they would buy this product. You may also assume that the total number of people who could be surveyed is large enough that different peoples responses are independent.]

   The number of people who say they would buy the product is binomially distributed with $n = 300$ and $p = 0.4$. We can approximate this as normally distributed with mean $300 \times 0.4 = 120$ and variance $300 \times 0.4 \times 0.6 = 72$. The proportion X who say they would buy the product is therefore normally distributed with mean 0.4 and variance 0.0008. Let $Z = \frac{X-0.4}{\sqrt{0.0008}}$, so $Z \sim N(0,1)$. We have $P(0.39 < X < 0.41) = P(-\frac{0.01}{\sqrt{0.0008}} < Z < \frac{0.01}{\sqrt{0.0008}}) = \Phi(\frac{0.01}{\sqrt{0.0008}}) - \Phi(-\frac{0.01}{\sqrt{0.0008}}) = 2\Phi(\frac{0.01}{\sqrt{0.0008}}) - 1$. We have $\frac{0.01}{\sqrt{0.0008}} = 0.35$, so $\Phi(\frac{0.01}{\sqrt{0.0008}}) = 0.6368$, so we have $P(0.39 < X < 0.41) = 0.27$.

   (b) Find the smallest interval such that the probability that the true proportion is in this interval is at least 95%.

   The smallest interval is of the form $-a < Z < a$ for some a. The probability of lying in this interval is $2\Phi(a) - 1 = 0.95$, so we must have $\Phi(a) = 0.975$. From the table, we see that this gives $a = 1.96$. We therefore have $-1.96 < Z < 1.96$ or $0.4-1.96\sqrt{0.0008} < X < 0.4+1.96\sqrt{0.0008}$ or $0.345 < X < 0.455$. So X is between 34.5% and 45.5% with probability 0.95.

   (c) How many people do they need to survey so that the probability that their estimate is within 1% of the true value is at least 95%?

   They want a 95% probability that their estimate X is between 0.39 and 0.41. Since we know the expected value of X is 0.4, we want the standard deviation of X to satisfy $\frac{0.01}{\sigma} = 1.96$. We know that $\sigma^2 = \frac{0.24}{n}$, so we want $\frac{0.24}{n} = \left(\frac{0.01}{1.96}\right)^2$, so $n = 0.24^2 = 9219.84$.

   (d) [bonus] How many people do they need to survey to be sure that whatever the true proportion is, their estimate is within 1% of it with probability at least 0.95.

   Similarly to the previous question, they need the variance of X to be at most $\frac{1}{196^2}$, but here the variance of X is $\frac{p(1-p)}{n}$, where p is the true proportion of people who would buy their product. Therefore, to ensure that $\frac{p(1-p)}{n} \leqslant \frac{1}{196^2}$ for any value of p, we observe that the maximum value of $p(1-p)$ is 0.25 (when $p = 0.5$) and so we need to find n such that $\frac{0.25}{n} \leqslant \frac{1}{196^2}$, or $n = 0.25 \times 196^2 = 9604$.

6. *A company makes computers. The lifetime of the computers it makes (in years) is exponentially distributed with parameter 0.3. You upgrade your computer every 3 years. What is the probability that a computer from this company breaks down before you are ready to upgrade it?*

   This is the probability that the lifetime of the computer is at most 3 years, which is given by $1 - e^{-3\times 0.3} = 0.5934$.

7. *The number of customers a company has on a given day is a Poisson random variable with parameter 3.6. What is the probability that the company receives exactly 5 customers on a given day?*

   Let N be the number of customers. We have $P(N = 5) = e^{-3.6}\frac{3.6^5}{5!} = 0.1377$.

8. *A company produces 80,000,000 scratchcards, which it will sell for \$1 each. The scratchcards offer the following cash prizes:*

Prize	*Number of cards with this prize*
0	*78,000,000*
\$10	*1,840,000*
\$100	*145,000*
\$1,000	*13,600*
\$10,000	*1,385*
\$1,000,000	*15*

   What is the expected value and variance for the prize of a scratch card?

   Let X be the prize on a scratchcard. We have $\mathbb{E}(X) = 10 \times \frac{184}{8000} + 100 \times \frac{145}{80000} + 1000 \times \frac{136}{800000} + 10000 \times \frac{1375}{80000000} + 1000000 \times \frac{25}{80000000} = \frac{}{8000} = \frac{7535}{8000} = 0.9419$.