MATH/STAT 3360, Probability
FALL 2014
Toby Kenney
Sample Final Examination

This Sample examination has more questions than the actual final, in order to cover a wider range of questions. Estimated times are provided after each question to help your preparation.

1. On average, in a certain course, 20% of students get an 'A' and 15% students get a 'B'. There are 25 students taking the course one year. What is the probability that 5 students get an 'A' and 5 students get a 'B'? [5 mins]

2. (R, Θ) have joint density function $f_{(R,\Theta)}(r, \theta) = \frac{1}{2\pi}e^{-r}$. Let $X = R\cos\Theta$ and $Y = R\sin\Theta$.

   (a) What is the joint density function of (X, Y)? [15 mins]

   (b) Are X and Y independent? [5 mins]

3. (R, Θ) have joint density function $f_{(R,\Theta)}(r, \theta) = \frac{1}{2\pi}re^{-\frac{r^2}{2}}$. Let $X = R\cos\Theta$ and $Y = R\sin\Theta$.

   (a) What is the joint density function of (X, Y)? [15 mins]

   (b) Are X and Y independent? [5 mins]

4. Random variables X and Y have joint density function $f_{X,Y}(x, y) = \frac{2}{5\pi}e^{-\frac{5x^2+5y^2-6xy}{10}}$.

   (a) What is the joint density function of $Z = X + Y$ and $W = X - Y$? [15 mins]

   (b) Are W and Z independent? [5 mins]

   (c) What is $\mathrm{Cov}(X, Y)$? [Hint: consider $\mathrm{Var}(Z) - \mathrm{Var}(W)$.] [10 mins]

5. Let X and Y are independent normal random variables with mean 0 and variances σ_1^2 and σ_2^2 respectively.

   (a) What is the joint density function of X and Y? [5 mins]

   (b) If (R, Θ) are the polar coordinates of (X, Y) (that is, $R = \sqrt{X^2 + Y^2}$ and Θ is the solution to $R\cos\Theta = X$ and $R\sin\Theta = Y$) then what is the joint density function of R and Θ? [15 mins]

   (c) Calculate the marginal density function of Θ. [10 mins]

6. X is normally distributed with mean 0 and variance 1. Y is independent of X and has probability density functions

$$f_Y(y) = \frac{1}{\pi(1 + y^2)}$$

(a) Find the joint density function of $Z = 2X + Y$ and $W = X + 2Y$. [15 mins]

(b) Are Z and W independent? [5 mins]

7. Let X have an exponential distribution with parameter λ. Suppose that given $X = x$, Y is normally distributed with mean x and variance σ^2.

   (a) What is the joint density function of X and Y? [5 mins]

   (b) What is the covariance of X and Y? [10 mins]

   (c) Conditional on $Y = 3$, what is the density function of X? [You do not need to calculate the constant factor.] [5 mins]

8. X is normally distributed with mean 0 and variance 1. Y is independent of X and has probability density functions

$$f_Y(y) = \frac{\lambda}{2} e^{-\lambda|y|}$$

   for some constant $\lambda > 0$. Find the joint density function of $Z = X + Y$ and $W = X + 3Y$. [10 mins]

9. Let X have a Poisson distribution with parameter λ, and given that $X = n$, let Y have a binomial distribution with parameters n and p.

   (a) What is the joint probability mass function of X and Y? [5 mins]

   (b) Show that the marginal distribution of Y is a Poisson distribution with parameter λp. [10 mins]

   (c) What is the conditional distribution of X conditional on $Y = i$? [10 mins]

10. Let X have a normal distribution with mean 0 and standard deviation 1. Suppose that given $X = x$, Y is normally distributed with mean $2x - 3$ and variance $\frac{\sigma^2}{x}$.

    (a) What is the joint density function of X and Y? [5 mins]

    (b) What is the covariance of X and Y? [10 mins]

11. X is normally distributed with mean 2 and standard deviation 3. Y is normally distributed with mean 4 and standard deviation 4. What is $P(Y > X)$?

    (a) If X and Y are independent? [5 mins]

    (b) If X, Y have a multivariate normal distribution, and $\mathrm{Cov}(X, Y) = 2$? [10 mins]

12. X is normally distributed with mean 5 and standard deviation 1. Y is independent and normally distributed with mean 2 and standard deviation 3. What is $P(Y > X)$? [5 mins]

13. X is normally distributed with mean 3 and standard deviation 5. Y is normally distributed with mean 7 and standard deviation 12. What is $P(Y > X)$?

    (a) If X and Y are independent? [5 mins]

    (b) If X, Y have a multivariate normal distribution, and $\text{Cov}(X, Y) = 24$? [10 mins]

14. Let X and Y be independent uniform random variables on intervals $[2, 4]$ and $[3, 4]$ respectively. Calculate the probability density function of $X+Y$. [10 mins]

15. Let X and Y be independent exponential random variables with parameters 2 and 3. What is the probability that $X + Y > 1$. [10 mins]

16. If X is exponentially distributed with parameter λ_1 and Y is independent of X and exponentially distributed with parameter λ_2, what is the probability density function of $X + Y$? [Assume $\lambda_1 \neq \lambda_2$.] [10 mins]

17. In a class of 100 students, the professor wants to determine what proportion of the students can answer a simple calculus question. The professor decides to test a random sample of 10 students. In fact 35 of the students could answer the question.

    (a) What is the expected number of students sampled who answer the question correctly? [5 mins]

    (b) What is the variance of the number of correct answers if the students are sampled with replacement, i.e. one student could be tested more than once? [10 mins]

    (c) What is the variance of the number of correct answers if the 10 students sampled must be 10 different students? [10 mins]

18. An ecologist is collecting butterflies. She collects a total of 50 butterflies. There are a total of 24 species of butterfly, and each butterfly is equally likely to be any of the species.

    (a) What is the expected number of species of butterflies she collects? [10 mins]

    (b) What is the variance of the number of species of butterflies she collects? [10 mins]

    (c) Using the one-sided Chebyshev inequality, find a bound on the probability that she collects all 24 species. [Hint: The Chebyshev inequality will give the probability of collecting at least 24 species.] [10 mins]

19. An ecologist is collecting snails. She collects a total of 40 snails. There are a total of 18 species of snail, and each snail is equally likely to be any of the species.

(a) What is the expected number of species of snails she collects? [10 mins]

(b) What is the variance of the number of species of snails she collects? [10 mins]

(c) Using the one-sided Chebyshev inequality, find a bound on the probability that she collects all 18 species. [Hint: The Chebyshev inequality will give the probability of collecting at least 18 species.] [10 mins]

20. An ecologist is collecting beetles. She collects a total of 16 beetles. There are a total of 15 species of beetle, and each beetle is equally likely to be any of the species.

    (a) What is the expected number of species of beetles she collects? [10 mins]

    (b) What is the variance of the number of species of beetles she collects? [10 mins]

    (c) Using the (two-sided) Chebyshev inequality, find a lower bound on the probability that the number of species she collects is between 8 and 12 inclusive. [Hint: If it is not between 8 and 12, it must be either at least 13, or at most 7.] [10 mins]

21. Consider the following experiment: Toss 14 fair coins. For each toss that results in a head, roll a fair (6-sided) die. Take the sum of the numbers rolled on all these dice. What is the expected outcome? [5 mins]

22. Consider the following experiment: Roll 100 fair (6-sided) dice. For each 3 rolled, toss a fair coin, and for each 6 rolled, toss 2 fair coins. Count the number of heads on all coins tossed. What is the expected outcome? [5 mins]

23. Consider the following experiment: Roll a fair (6-sided) die. If the result n is odd, roll $2n$ fair dice and take the sum of the numbers. If n is even, toss $2n^2 - n$ fair coins and count the number of heads. What is the expected outcome? [5 mins]

24. The point (X, Y) is uniformly distributed on the set

    $$\{(x, y) | x + y < 3, y > -1, x > -2\}$$

    .

    (a) What is the conditional expectation of X given that $Y = 2$? [5 mins]

    (b) What is $\mathrm{Cov}(X, Y)$? [10 mins]

25. An insurance company finds that of its home insurance policies, only 0.5% result in a claim. Of the policies that result in a claim, the expected amount claimed is \$40,000, and the standard deviation of the amount claimed is \$100,000.

(a) What are the expected amount claimed, and the variance of the amount claimed for any policy? [If a policy does not result in a claim, the amount claimed is \$0. Hint: for the variance, try to work out $\mathbb{E}(X^2)$.] [10 mins]

(b) If the company sets its premium to \$430, but has administrative costs of \$200 for each policy, and the company sells 500,000 policies, what is the approximate probability that the company is unable to pay out all the claims from the premiums collected? [10 mins]

26. A car insurance company finds that of its claims, 70% are for accidents, and 30% are for thefts. The theft claims are all for \$20,000, while for the accident claims claim, the expected amount claimed is \$15,000, and the standard deviation of the amount claimed is \$30,000.

    (a) What are the expected amount claimed, and the standard deviation of the amount claimed for any claim? [10 mins]

    (b) If the company has \$16,700,000 available to cover claims, and receives 1000 claims, what is the probability that it is unable to cover the claims made? [10 mins]

27. A truncated standard normal random variable has probability density function

$$f_X(x) = \begin{cases} \frac{e^{-\frac{x^2}{2}}}{\int_a^b e^{-\frac{x^2}{2}}} & \text{if } a \leqslant x \leqslant b \\ 0 & \text{otherwise} \end{cases}$$

    That is, it is a normal distribution conditional on lying in the interval $[a, b]$.

    The expectation of a truncated standard normal is given by

$$E(X) = \frac{e^{-\frac{a^2}{2}} - e^{-\frac{b^2}{2}}}{\sqrt{2\pi}(\Phi(b) - \Phi(a))}$$

    An investment advisor tells you that the amount by which the market will increase next year is normally distributed with mean 10% and standard deviation 10%. The bank offers an investment product which pays the value of the market, minus a 2% fee if the market increases by between 2% and 32%. If the market rises by less than 2%, then it pays out your original investment; if the market rises by more than 32%, it pays out your original investment plus 30%. What is the expected value of this investment? [Hint: divide into the three cases where the market increases by less than 2%, the market increases by between 2% and 32%, and the market increases by over 32%.] [15 mins]

28. You are considering an investment. You would originally invest \$1,000, and every year, the investment will either increase by 50% with probability 0.6 or decrease by 30% with probability 0.4. You plan to use the

investment after 15 years. What is the expected value of the investment after 15 years? [10 mins]

29. If X is exponentially distributed with parameter λ_1 and Y is independent of X and normally distributed with mean μ and variance σ^2.

    (a) find the moment generating function of $X - Y$. [10 mins]

    (b) Use the Chernoff bound with $t = \frac{-2\mu}{\sigma^2}$ to obtain a *lower* bound on the probability that $X > Y$. [10 mins]

30. Let X be exponentially distributed with parameter λ_1 and Y be independent of X and exponentially distributed with parameter λ_2, where $\lambda_1 > \lambda_2$.

    (a) find the moment generating function of $X - Y$. [10 mins]

    (b) Use the Chernoff bound with $t = \frac{\lambda_1 - \lambda_2}{2}$ to obtain an *upper* bound on the probability that $X > Y$. [10 mins]

31. If X is exponentially distributed with parameter λ_1 and Y is exponentially distributed with parameter λ_2, where $\lambda_2 > \lambda_1$:

    (a) find the moment generating function of $X - Y$. [10 mins]

    (b) Use the Chernoff bound with $t = \frac{\lambda_1 - \lambda_2}{2}$ to obtain a *lower* bound on the probability that $X > Y$. [10 mins]

32. If X is uniformly distributed on [-1,4], and Y is uniformly distributed on the interval $[0, 3]$, find the moment generating function of $X - Y$. [10 mins]

33. A company is planning to run an advertising campaign. It estimates that the number of customers it gains from the advertising campain will be approximately normally distributed with mean 3,000 and standard deviation 300. It also estimates that the amount spent by each customer has expected value $50 and standard deviation $30.

    (a) Assuming the number of customers is large enough, what is the approximate distribution of the average amount spent per customer, conditional on the number of new customers being n. [5 mins]

    (b) What is the joint density function for the number of new customers and the average amount spent per customer? [10 mins]

34. A gambler is playing a slot machine in a casino. The gambler continues to bet $1 each time. The machine has the following payouts:

Payout	Probability
$1	0.4
$10	0.03
$100	0.002
$1000	0.00005

How many times does the gambler have to play before the probability of his having more money than he started with is less than 1%? [10 mins]

35. An investor's annual profit has an expectation of \$2,000 and a variance of 100,000,000. (Profits in different years are independent and identically distributed.)

    (a) After n years, for large n, what is the approximate distribution of the investors average annual profit? [5 mins]

    (b) After how many years is the probability that the investor has made an overall loss during those years less than 0.001? [10 mins]

    (c) If at any point, the investor has made a total loss of over \$100,000, he must pay a fine. After how many years is the danger that he will have to pay this fine greatest, and what is the danger after this many years [assuming the approximation from (a) is valid]? [10 mins]

36. The number of visitors to a particular website on a given day is approximately normally distributed with mean 12000 and variance 2000^2. A company is considering placing an advertisement on this website. It predicts that each visitor to the website will order its product with probability 0.02, and that all visitors to the website act independently.

    (a) What is the expectation and variance of the number of orders the company receives? [10 mins]

    (b) Using a normal approximation, what is the probability that the number of orders the company receives is more than 300? [5 mins]