MATH/STAT 3360, Probability
FALL 2014
Toby Kenney
Sample Final Examination
Model Solutions

This Sample examination has more questions than the actual final, in order to cover a wider range of questions. Estimated times are provided after each question to help your preparation.

1. *On average, in a certain course, 20% of students get an 'A' and 15% students get a 'B'. There are 25 students taking the course one year. What is the probability that 5 students get an 'A' and 5 students get a 'B'? [5 mins]*

   This is a multinomial distribution. The probability that 5 students get an 'A' and 5 students get a 'B' is therefore $\binom{25}{5,5,15}0.2^5 0.15^5 0.65^{15} = 0.0313$.

2. *(R,Θ) have joint density function $f_{(R,\Theta)}(r,\theta) = \frac{1}{2\pi}e^{-r}$. Let $X = R\cos\Theta$ and $Y = R\sin\Theta$.*

   (a) What is the joint density function of (X,Y)? [15 mins]

   The Jacobian matrix is

   $$\begin{pmatrix} -r\sin\theta & \cos\theta \\ r\cos\theta & \sin\theta \end{pmatrix}$$

   so its determinant is $-r\sin^2\theta - r\cos^2\theta = -r$.

   We therefore have $f_{X,Y}(r\cos\theta, r\sin\theta) = \frac{f_{R,\Theta}(r,\theta)}{r}$ (where r is always positive, so there is no need to take its absolute value).

   This gives $f_{X,Y}(r\cos\theta, r\sin\theta) = \frac{e^{-r}}{2\pi r}$. We now write $r = \sqrt{x^2+y^2}$ to get

   $$f_{X,Y}(x,y) = \frac{e^{-\sqrt{x^2+y^2}}}{2\pi\sqrt{x^2+y^2}}$$

   (b) Are X and Y independent? [5 mins]

   X and Y are not independent, because this probability density function cannot be written as a product of a function of X and a function of Y.

3. *(R,Θ) have joint density function $f_{(R,\Theta)}(r,\theta) = \frac{1}{2\pi}re^{-\frac{r^2}{2}}$. Let $X = R\cos\Theta$ and $Y = R\sin\Theta$.*

   (a) What is the joint density function of (X,Y)? [15 mins]

   $\frac{\partial X}{\partial R} = \cos\Theta$ $\frac{\partial Y}{\partial R} = \sin\Theta$ $\frac{\partial X}{\partial\Theta} = -R\sin\Theta$ $\frac{\partial Y}{\partial\Theta} = R\cos\Theta$, so the Jacobian of X,Y with respect to $R\Theta$ has determinant $R(\cos^2\Theta+\sin^2\Theta) = R$. Therefore, $f_{X,Y}(x,y) = \frac{1}{2\pi}\frac{re^{-\frac{r^2}{2}}}{r} = \frac{1}{2\pi}e^{-\frac{x^2+y^2}{2}}$.

(b) Are X and Y independent? [5 mins]

The joint density function is $f_{X,Y}(x,y) = \frac{1}{2\pi}e^{-\frac{x^2+y^2}{2}} = \frac{1}{2\pi}e^{-\frac{x^2}{2}}e^{-\frac{y^2}{2}}$ which is a product of a function of x and a function of y, so X and Y are independent.

4. *Random variables X and Y have joint density function* $f_{X,Y}(x,y) = \frac{2}{5\pi}e^{-\frac{5x^2+5y^2-6xy}{10}}$.

   (a) What is the joint density function of $Z = X+Y$ *and* $W = X-Y$*? [15 mins]*

$$\frac{\partial z}{\partial x} = 1 \qquad \frac{\partial z}{\partial y} = 1$$
$$\frac{\partial w}{\partial x} = 1 \qquad \frac{\partial w}{\partial y} = -1$$

   So we get that $2dxdy = dzdw$. Therefore, the joint density function of (Z,W) is given by $f_{Z,W}(z,w) = \frac{1}{2}\frac{2}{5\pi}e^{-\frac{5x^2+5y^2-6xy}{10}}$. Now $x = \frac{1}{2}(z+w)$ and $y = \frac{1}{2}(z-w)$, so $5x^2+5y^2-6xy = \frac{5}{4}(2z^2+2w^2) - \frac{6}{4}(z^2-w^2) = z^2+4w^2$, so $f_{Z,W}(z,w) = \frac{1}{5\pi}e^{-\frac{z^2+4w^2}{10}}$.

   (b) Are W and Z independent? [5 mins]

   $f_{Z,W}(z,w) = \frac{1}{5\pi}e^{-\frac{z^2+4w^2}{10}} = \frac{1}{5\pi}e^{-\frac{z^2}{10}}e^{-\frac{4w^2}{10}}$ is a product of a function of z and a function of w, so Z and W are independent.

   (c) What is $\mathrm{Cov}(X,Y)$*? [Hint: consider* $\mathrm{Var}(Z) - \mathrm{Var}(W)$*.] [10 mins]*

   Recall that $\mathrm{Var}(X+Y) = \mathrm{Var}(X)+\mathrm{Var}(Y)+2\,\mathrm{Cov}(X,Y)$, and $\mathrm{Var}(X-Y) = \mathrm{Var}(X)+\mathrm{Var}(Y)-2\,\mathrm{Cov}(X,Y)$. Therefore, $\mathrm{Var}(X+Y)-\mathrm{Var}(X-Y) = 4\,\mathrm{Cov}(X,Y)$. Now from (a) and (b), we see that $\mathrm{Var}(X+Y) = \mathrm{Var}(Z) = 5$, and $\mathrm{Var}(X-Y) = \mathrm{Var}(W) = 1.25$, so $4\,\mathrm{Cov}(X,Y) = 5 - 1.25 = 3.75$, so $\mathrm{Cov}(X,Y) = 0.9375$.

5. *Let X and Y be independent normal random variables with mean 0 and variances* σ_1^2 *and* σ_2^2 *respectively.*

   (a) What is the joint density function of X and Y? [5 mins]

   $f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_1\sigma_2}e^{-\frac{\sigma_2^2x^2+\sigma_1^2y^2}{2(\sigma_1^2\sigma_2^2)}}$.

   (b) If (R,Θ) *are the polar coordinates of* (X,Y) *(that is,* $R = \sqrt{X^2+Y^2}$ *and* Θ *is the solution to* $R\cos\Theta = X$ *and* $R\sin\Theta = Y$*) then what is the joint density function of R and* Θ*? [15 mins]*

   If $x = r\cos\theta$ and $y = r\sin\theta$, then

$$\frac{\partial x}{\partial r} = \cos\theta$$
$$\frac{\partial x}{\partial \theta} = -r\sin\theta$$
$$\frac{\partial y}{\partial r} = \sin\theta$$
$$\frac{\partial y}{\partial \theta} = r\cos\theta$$

Therefore, we get that $dxdy = rdrd\theta$, so that the joint density function of (R,Θ) is $f_{R,\Theta}(r,\theta) = \frac{r}{2\pi\sigma_1\sigma_2}e^{-\frac{\sigma_2^2x^2+\sigma_1^2y^2}{2(\sigma_1^2\sigma_2^2)}} = \frac{r}{2\pi\sigma_1\sigma_2}e^{-\frac{r^2(\sigma_2^2\cos^2\theta+\sigma_1^2\sin^2\theta)}{2(\sigma_1^2\sigma_2^2)}}$.

(c) Calculate the marginal density function of Θ. [10 mins]

The marginal density of Θ is given by integrating R, so $f_\Theta(\theta) = \int_0^\infty \frac{r}{2\pi\sigma_1\sigma_2}e^{-\frac{r^2(\sigma_2^2\cos^2\theta+\sigma_1^2\sin^2\theta)}{2(\sigma_1^2\sigma_2^2)}}dr = \frac{1}{2\pi\sigma_1\sigma_2}\left[-\frac{e^{-\frac{r^2(\sigma_2^2\cos^2\theta+\sigma_1^2\sin^2\theta)}{2\sigma_1^2\sigma_2^2}}}{\frac{(\sigma_2^2\cos^2\theta+\sigma_1^2\sin^2\theta)}{\sigma_1^2\sigma_2^2}}\right]_0^\infty = \frac{\sigma_1\sigma_2}{2\pi(\sigma_2^2\cos\theta+\sigma_1^2\sin\theta)}$.

6. *X is normally distributed with mean 0 and variance 1. Y is independent of X and has probability density functions*

$$f_Y(y) = \frac{1}{\pi(1+y^2)}$$

(a) Find the joint density function of $Z = 2X+Y$ and $W = X+2Y$. [15 mins]

The joint density function of X and Y is $\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\frac{1}{\pi(1+y^2)}$. The Jacobian matrix for the change of variable is $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, which has determinant $2\times 2 - 1\times 1 = 3$. This gives

$$f_{Z,W}(x,y) = \frac{e^{-\frac{x^2}{2}}}{3\pi\sqrt{2\pi}(1+y^2)}$$

We need to rewrite this in terms of Z and W. Solving the equations for X and Y gives $X = \frac{2Z-W}{3}$ and $Y = \frac{2W-Z}{3}$, so we get

$$f_{Z,W}(z,w) = \frac{e^{-\frac{(2z-w)^2}{18}}}{3\pi\sqrt{2\pi}\left(1+\left(\frac{2w-z}{3}\right)^2\right)} = \frac{3e^{-\frac{(2z-w)^2}{18}}}{\pi\sqrt{2\pi}\left(9+(2w-z)^2\right)}$$

(b) Are Z and W independent? [5 mins]

The joint density function is not a product of a function of z and a function of w, so Z and W are not independent.

7. *Let X have an exponential distribution with parameter λ. Suppose that given $X = x$, Y is normally distributed with mean x and variance σ^2.*

   (a) What is the joint density function of X and Y? [5 mins]

   We have that $f_{X,Y}(x,y) = f_X(x) f_{Y|X}(y) = \frac{\lambda}{\sqrt{2\pi}\sigma} e^{-\lambda x} e^{-\frac{(y-x)^2}{2\sigma^2}}$.

   (b) What is the covariance of X and Y? [10 mins]

   We know that $\mathbb{E}(X) = \frac{1}{\lambda}$. On the other hand, we have that $\mathbb{E}(Y|X = x) = x$, so $\mathbb{E}(Y) = \mathbb{E}(\mathbb{E}(Y|X)) = \mathbb{E}(X) = \frac{1}{\lambda}$.

   Finally, we consider $\mathbb{E}(XY|X = x) = x\mathbb{E}(Y|X = x) = x^2$, so $\mathbb{E}(XY) = \mathbb{E}(X^2) = \frac{2}{\lambda^2}$. Therefore, we have $\mathrm{Cov}(X,Y) = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$.

   (c) Conditional on $Y = 3$, what is the density function of X? [You do not need to calculate the constant factor.] [5 mins]

   Conditional on $Y = 3$, we get $f_{X|Y}(x|y = 3) = Ce^{-\lambda x} e^{-\frac{(x-y)^2}{2\sigma^2}} = Ce^{-\frac{x^2+2(y-\lambda\sigma^2)x+y^2)}{2\sigma^2}}$, provided $x > 0$ and $f_{X|Y}(x|y = 3) = 0$ for $x < 0$. [So X has a truncated normal distribution with mean $y - \lambda\sigma^2$, and variance σ^2.]

8. *X is normally distributed with mean 0 and variance 1. Y is independent of X and has probability density functions*

   $$f_Y(y) = \frac{\lambda}{2} e^{-\lambda|y|}$$

   for some constant $\lambda > 0$. Find the joint density function of $Z = X + Y$ and $W = X + 3Y$. [10 mins]

   The joint density function of X and Y is

   $$f_{X,Y}(x,y) = \frac{\lambda}{2\sqrt{2\pi}} e^{-\frac{x^2}{2}} e^{-\lambda|y|}$$

   We have that the Jacobian matrix for the function sending (x,y) to (z,w) is

   $$\begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}$$

   so its determinant is $1 \times 3 - 1 \times 1 = 2$. We therefore have $f_{Z,W}(z,w) = \frac{f_{X,Y}(x,y)}{2}$, furthermore, by solving the equations we have $x = \frac{3z-w}{2}$ and $y = \frac{w-z}{2}$, so we get

   $$f_{Z,W}(z,w) = \frac{\lambda}{4\sqrt{2\pi}} e^{-\frac{(3z-w)^2}{8} - \frac{\lambda|w-z|}{2}}$$

9. *Let X have a Poisson distribution with parameter λ, and given that $X = n$, let Y have a binomial distribution with parameters n and p.*

(a) What is the joint probability mass function of X and Y? [5 mins]

We have $P(X = n, Y = i) = e^{-\lambda}\frac{\lambda^n}{n!}\binom{n}{i}p^i(1-p)^{n-i}$, whenever $i \leqslant n$.

[Alternatively, we can write this as $P(X = n, Y = i) = e^{-\lambda}\frac{\lambda^n}{i!(n-i)!}p^i(1-p)^{n-i} = e^{-\lambda}\frac{(\lambda p)^i}{i!}\frac{(\lambda(1-p))^{n-i}}{(n-i)!}$.]

(b) Show that the marginal distribution of Y is a Poisson distribution with parameter λp. [10 mins]

The marginal distribution of Y is given by

$$P(Y = i) = \sum_{n=i}^{\infty}\left(e^{-\lambda}\frac{(\lambda p)^i}{i!}\frac{(\lambda(1-p))^{n-i}}{(n-i)!}\right)$$

We substitute $m = n - i$ in the sum to get

$$P(Y = i) = \sum_{m=0}^{\infty}\left(e^{-\lambda}\frac{(\lambda p)^i}{i!}\frac{(\lambda(1-p))^m}{m!}\right) = e^{-\lambda p}\frac{(\lambda p)^i}{i!}e^{-\lambda(1-p)}\sum_{m=0}^{\infty}\frac{((\lambda(1-p))^m)}{m!} = e^{-\lambda p}\frac{(\lambda p)^i}{i!}$$

(c) What is the conditional distribution of X conditional on $Y = i$? [10 mins]

We have $P(X = n|Y = i) = \frac{P(X=n,Y=i)}{P(Y=i)} = \frac{\left(e^{-\lambda}\frac{(\lambda p)^i}{i!}\frac{(\lambda(1-p))^{n-i}}{(n-i)!}\right)}{e^{-\lambda p}\frac{(\lambda p)^i}{i!}} = e^{-\lambda(1-p)}\frac{(\lambda(1-p))^{n-i}}{(n-i)!}$

[So the conditional distribution of X is i plus a Poisson random variable with parameter $\lambda(1-p)$.]

10. *Let X have a normal distribution with mean 0 and standard deviation 1. Suppose that given $X = x$, Y is normally distributed with mean $2x - 3$ and variance $\frac{\sigma^2}{x}$.*

(a) What is the joint density function of X and Y? [5 mins]

The joint density function of X and Y is given by

$$f_{X,Y}(x, y) = f_X(x)f_{Y|X}(y|x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\frac{\sqrt{x}}{\sqrt{2\pi}\sigma}e^{-\frac{x(y-2x+3)^2}{2\sigma^2}} = \frac{\sqrt{x}}{2\pi\sigma}e^{-\frac{x^2}{2}-\frac{x(y-2x+3)^2}{2\sigma^2}}$$

(b) What is the covariance of X and Y? [10 mins]

The covariance of X and Y is given by $\mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$. We also have $\mathbb{E}(XY) = \mathbb{E}(\mathbb{E}(XY|X))$. Conditional on $X = x$, we have $XY = xY$, and Y is normally distributed with mean $2x - 3$, so we have $\mathbb{E}(XY|X = x) = x(2x-3)$, so $\mathbb{E}(XY) = \mathbb{E}(X(2X-3)) = 2\mathbb{E}(X^2)-3\mathbb{E}(X) = 2\times1-3\times0 = 2$. Therefore, $\mathrm{Cov}(X, Y) = 2$.

11. *X is normally distributed with mean 2 and standard deviation 3. Y is normally distributed with mean 4 and standard deviation 4. What is $P(Y > X)$?*

    (a) If X and Y are independent? [5 mins]

    If X and Y are independent, then $Y - X$ is normally distributed with mean $4 - 2 = 2$ and variance $3^2 + 4^2 = 25$, so it has standard deviation 5. We want $P(Y - X > 0)$. Let $Z = \frac{Y-X-2}{5}$, then $Z \sim N(0,1)$, and we are looking for $P(Z > -0.4) = 1 - \Phi(-0.4) = \Phi(0.4) = 0.6554$.

    (b) If X, Y have a multivariate normal distribution, and $\mathrm{Cov}(X,Y) = 2$? [10 mins]

    Again we have $Y - X$ is normally distributed with mean 2, but now the variance is $3^2 + 4^2 - 2\,\mathrm{Cov}(X,Y) = 21$, so $P(Y - X > 0) = \Phi\left(\frac{2}{\sqrt{21}}\right) = \phi(0.44) = 0.6700$.

12. *X is normally distributed with mean 5 and standard deviation 1. Y is independent and normally distributed with mean 2 and standard deviation 3. What is $P(Y > X)$? [5 mins]*

    $Y > X$ is equivalent to $X - Y < 0$. We know that $-Y$ is normally distributed with mean -2 and standard deviation 3, so $X - Y$ is normally distributed with mean 3 and variance $1^2 + 3^2 = 10$. Therefore $P(X - Y < 0) = \Phi\left(\frac{0-3}{\sqrt{10}}\right) = 1 - \Phi\left(\frac{3}{\sqrt{10}}\right) = 1 - \Phi(0.95) = 0.1711$.

13. *X is normally distributed with mean 3 and standard deviation 5. Y is normally distributed with mean 7 and standard deviation 12. What is $P(Y > X)$?*

    (a) If X and Y are independent? [5 mins]

    If X and Y are independent, then $Y - X$ is normally distributed with mean 4 and variance $5^2 + 12^2 = 169$, so $P(Y > X) = P(Y - X) > 0 = 1 - \Phi\left(\frac{-4}{13}\right) = \Phi\left(\frac{4}{13}\right) = \Phi(0.31) = 0.6217$.

    (b) If X, Y have a multivariate normal distribution, and $\mathrm{Cov}(X,Y) = 24$? [10 mins]

    If $\mathrm{Cov}(X,Y) = 24$ then we have $\mathrm{Var}(Y - X) = \mathrm{Var}(X) + \mathrm{Var}(Y) - 2\,\mathrm{Cov}(X,Y) = 25 + 144 - 48 = 121$, so $P(Y > X) = P(Y - X > 0) = \Phi\left(\frac{4}{11}\right) = \Phi(0.36) = 0.6406$.

14. *Let X and Y be independent uniform random variables on intervals $[2,4]$ and $[3,4]$ respectively. Calculate the probability density function of $X + Y$. [10 mins]*

    We calculate this by convolution:

    $$f_{X+Y}(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)$$

    This integrand $f_x(x) f_Y(z-x)$ is zero unless:

- $5 \leqslant z \leqslant 6$, and $2 \leqslant x \leqslant z-3$.
- $6 \leqslant z \leqslant 7$, and $z-4 \leqslant x \leqslant z-3$.
- $7 \leqslant z \leqslant 8$, and $z-4 \leqslant x \leqslant 4$.

In these cases, we have $f_X(x) = 0.5$ and $f_Y(y) = 1$, so that

$$f_{X+Y}(z) = \begin{cases} 0 & \text{if } z < 5 \\ \frac{z-5}{2} & \text{if } 5 \leqslant z \leqslant 6 \\ \frac{1}{2} & \text{if } 6 \leqslant z \leqslant 7 \\ \frac{8-z}{2} & \text{if } 7 \leqslant z \leqslant 8 \\ 0 & \text{if } z \geqslant 8 \end{cases}$$

15. *Let X and Y be independent exponential random variables with parameters 2 and 3. What is the probability that $X+Y > 1$. [10 mins]*

    $P(X+Y) > 1 = P(X > 1) + \int_0^1 \int_{1-x}^{\infty} 6e^{-2x-3y} dy dx$. However, we know that $\int_a^{\infty} e^{-3y} dy = \frac{e^{-3a}}{3}$, so we get $P(X+Y) > 1 = P(X > 1) + \int_0^1 2e^{-2x} e^{-3(1-x)} dx = e^{-2} + \int_0^1 2e^{x-3} dx = e^{-2} + 2e^{-3}(e^1 - e^0) = 3e^{-2} - 2e^{-3} = 0.3064$.

16. *If X is exponentially distributed with parameter λ_1 and Y is independent of X and exponentially distributed with parameter λ_2, what is the probability density function of $X+Y$? [Assume $\lambda_1 \neq \lambda_2$.] [10 mins]*

    The probability density function of $X+Y$ is given by the convolution

$$\begin{aligned} f_{X+Y}(a) &= \int_0^a f_X(x) f_Y(a-x) dx = \int_0^a \lambda_1 \lambda_2 e^{-\lambda_1 x} e^{-\lambda_2 (a-x)} dx \\ &= \lambda_1 \lambda_2 e^{-\lambda_2 a} \int_0^a e^{(\lambda_2 - \lambda_1) x} dx = \frac{\lambda_1 \lambda_2}{(\lambda_2 - \lambda_1)} e^{-\lambda_2 a} \left(e^{(\lambda_2 - \lambda_1) a} - 1 \right) \\ &= \lambda_1 \lambda_2 \frac{e^{-\lambda_1 a} - e^{-\lambda_2 a}}{\lambda_2 - \lambda_1} \end{aligned}$$

17. *In a class of 100 students, the professor wants to determine what proportion of the students can answer a simple calculus question. The professor decides to test a random sample of 10 students. In fact 35 of the students could answer the question.*

    (a) What is the expected number of students sampled who answer the question correctly? [5 mins]

    Let the students who know the answer be numbers 1 to 35. Let X_i be the number of times each student is chosen. We have that $\mathbb{E}(X_i) = \frac{10}{100} = 0.1$, for each i. The number of students who answer correctly is $\sum_{i=1}^{35} X_i$, so the expected number is $\sum_{i=1}^{35} \mathbb{E}(X_i) = 35 \times 0.1 = 3.5$.

(b) What is the variance of the number of correct answers if the students are sampled with replacement, i.e. one student could be tested more than once? [10 mins]

Let $Y_1, \ldots, Y_{10}$ be indicator random variables for the events that the students tested know the answer. If the students are sampled with replacement, then we have that each Y_i is 1, and the Y_i are independent. We therefore have $\text{Var}(Y_1 + \cdots + Y_{10}) = \text{Var}(Y_1) + \cdots + \text{Var}(Y_{10})$. Each Y_i is 1 with probability 0.35 and zero with probability 0.65, so $\mathbb{E}(Y_i) = 0.35$ and $\mathbb{E}({Y_i}^2) = 0.35$, so $\text{Var}(Y_i) = 0.35 \times 0.65 = 0.2275$. Therefore, the variance of the number of correct answers is 2.275.

(c) What is the variance of the number of correct answers if the 10 students sampled must be 10 different students? [10 mins]

If the students must be different, then each X_i is either zero or one. Furthermore, for $i \neq j$, the probability that students i and j are both selected is $\frac{\binom{98}{8}}{\binom{100}{10}} = \frac{1}{110}$. We now have $\mathbb{E}((\sum_{i=1}^{35} X_i)^2) = \sum_{i=1}^{35} \sum_{j=1}^{35} \mathbb{E}(X_i X_j) = 35 \times 0.1 + \frac{34 \times 35}{110} = \frac{157.5}{11}$. We therefore have that the variance of the number of correct answers is $\frac{157.5}{11} - 3.5^2 = \frac{157.5 - 134.75}{11} = \frac{22.75}{11} = 2.068$.

18. *An ecologist is collecting butterflies. She collects a total of 50 butterflies. There are a total of 24 species of butterfly, and each butterfly is equally likely to be any of the species.*

    (a) What is the expected number of species of butterflies she collects? [10 mins]

    For each of the 24 species, the probability that she does not collect any butterflies of that species is $\left(\frac{23}{24}\right)^{50}$. If we let

    $$X_i = \begin{cases} 1 & \text{if she collects a butterfly of species } i \\ 0 & \text{otherwise} \end{cases}$$

    then we have that $\mathbb{E}(X_i) = 1 - \left(\frac{23}{24}\right)^{50}$. The total number of species she collects is $\sum_{i=1}^{24} X_i$, so the expected number of species she collects is $\mathbb{E}\left(\sum_{i=1}^{24} X_i\right) = \sum_{i=1}^{24} \mathbb{E}(X_i) = 24\left(1 - \left(\frac{23}{24}\right)^{50}\right) = 21.14$.

    (b) What is the variance of the number of species of butterflies she collects? [10 mins]

    From the above, we see that for $i \neq j$,

    $$X_i X_j = \begin{cases} 1 & \text{if she collects butterflies of both species } i \text{ and } j \\ 0 & \text{otherwise} \end{cases}$$

    and we see that $P(X_i X_j = 1) = 1 - P(X_i = 0) - P(X_j = 0) + P(X_i = X_j = X_0)$, and $P(X_i = X_j = 0) = \left(\frac{22}{24}\right)^{50}$, so that $\mathbb{E}(X_i X_j) = P(X_i X_j = 1) = 1 - 2\left(\frac{23}{24}\right)^{50} + \left(\frac{22}{24}\right)^{50} = 0.7747$. Let $S = \sum_{i=1}^{24} X_i$ be the number of species she collects. We know that $\mathbb{E}(S^2) = \sum_i \mathbb{E}(X_i^2) + \sum_{i \neq j} \mathbb{E}(X_i X_j) = 21.14 + 24 \times 23 \times 0.7747 = 448.80$, so $\text{Var}(S) = \mathbb{E}(S^2) - (\mathbb{E}(S))^2 = 1.811$.

(c) Using the one-sided Chebyshev inequality, find a bound on the probability that she collects all 24 species. [Hint: The Chebyshev inequality will give the probability of collecting at least 24 species.] [10 mins]

The one-sided Chebyshev inequality says that $P(S \geqslant \mathbb{E}(S) + a) \leqslant \frac{\sigma^2}{\sigma^2 + a^2}$, so $P(S = 24) = P(S \geqslant 24) \leqslant \frac{1.811}{1.811 + 2.86^2} = 0.1815$.

19. *An ecologist is collecting snails. She collects a total of 40 snails. There are a total of 18 species of snail, and each snail is equally likely to be any of the species.*

(a) What is the expected number of species of snails she collects? [10 mins]

Let $X_1, \ldots, X_{18}$ be indicator variables, to indicate that she did not collect any snails of species i. That is:

$$X_i = \begin{cases} 1 & \text{if she does not collect any snails of species } i \\ 0 & \text{if she collects at least one snail of species } i \end{cases}$$

Now we have $\mathbb{E}(X_i) = P(X_i) = \left(\frac{17}{18}\right)^{40}$, so the total number of species of snails she collects is $18 - X = 18 - (X_1 + \cdots + X_{18})$, and the $\mathbb{E}(X) = 18\left(1 - \left(\frac{17}{18}\right)^{40}\right) = 16.17$.

(b) What is the variance of the number of species of snails she collects? [10 mins]

The variance of the number of species she collects is the same as the variance of the number of species she doesn't collect. We have $\mathbb{E}(X^2) = \sum_{i=1}^{18} \sum_{j=1}^{18} \mathbb{E}(X_i X_j)$. If $i = j$, we know that $\mathbb{E}(X_i X_j) = \mathbb{E}(X_i) = \left(\frac{17}{18}\right)^{40}$, while if $i \neq j$, we have $\mathbb{E}(X_i X_j) = \left(\frac{16}{18}\right)^{40}$, so that $\mathbb{E}(X^2) = 18 \times 17 \times \left(\frac{16}{18}\right)^{40} + 18 \times \left(\frac{17}{18}\right)^{40}$ and $\text{Var}(X) = 18 \times 17 \times \left(\frac{16}{18}\right)^{40} + 18 \times \left(\frac{17}{18}\right)^{40} - 18^2 \times \left(\frac{17}{18}\right)^{80} = 1.234$.

(c) Using the one-sided Chebyshev inequality, find a bound on the probability that she collects all 18 species. [Hint: The Chebyshev inequality will give the probability of collecting at least 18 species.] [10 mins]

The one-sided Chebyshev inequality states that for $a > 0$, $P(X - \mathbb{E}(X) \geqslant a) \leqslant \frac{\text{Var}(X)}{\text{Var}(X) + a^2}$. In this case, it tells us that $P((18 - X) - \mathbb{E}(18 - X) \geqslant 18 - \mathbb{E}(18 - X)) \leqslant \frac{\text{Var}(X)}{\text{Var}(X) + (\mathbb{E}(X))^2} = 0.269$. So the probability that she collects all the species is at most 0.269.

20. *An ecologist is collecting beetles. She collects a total of 16 beetles. There are a total of 15 species of beetle, and each beetle is equally likely to be any of the species.*

(a) What is the expected number of species of beetles she collects? [10 mins]

Let X_i be 0 if she collects the ith species of beetle, and 0 otherwise. The total number of species she collects is therefore $15 - X_1 - \cdots - X_{16}$, so the expected number of species she collects is $15 - 15\mathbb{E}(X_i)$. Now $\mathbb{E}(X_i) = P(X_i = 1) = \left(\frac{14}{15}\right)^{16}$, so the expected number of species of beetles she collects is $15 - 15\left(\frac{14}{15}\right)^{16} = 10.0263$.

(b) What is the variance of the number of species of beetles she collects? [10 mins]

The variance of the number of species she collects is the same as the variance of the number of species she doesn't collect. That is $\mathrm{Var}(X_1 + \cdots + X_{15})$. We have $\mathbb{E}((X_1 + \cdots + X_{15})^2) = \mathbb{E}(\sum_{i=1}^{15}\sum_{j=1}^{15} X_iX_j) = \sum_{i=1}^{15}\sum_{j=1}^{15}\mathbb{E}(X_iX_j)$. We have $\mathbb{E}(X_iX_j) = \left(\frac{14}{15}\right)^{16}$ if $i = j$ and $\mathbb{E}(X_iX_j) = \left(\frac{13}{15}\right)^{16}$ if $i \neq j$, so $\mathbb{E}((X_1+\cdots+X_{15})^2) = 15\times 14\times\left(\frac{13}{15}\right)^{16} + 15\times\left(\frac{14}{15}\right)^{16}$, and $\mathrm{Var}(X_1+\cdots+X_{15}) = 15\times 14\times\left(\frac{13}{15}\right)^{16} + 15\times\left(\frac{14}{15}\right)^{16} - 15^2\times\left(\frac{14}{15}\right)^{32} = 1.5102$.

(c) Using the (two-sided) Chebyshev inequality, find a lower bound on the probability that the number of species she collects is between 8 and 12 inclusive. [Hint: If it is not between 8 and 12, it must be either at least 13, or at most 7.] [10 mins]

The Chebyshev inequality states that $P(|X - \mathbb{E}(X)| \geqslant a) \leqslant \frac{\mathrm{Var}(X)}{a^2}$. If the number of species she collects is between 7 and 13 exclusive, then we have $|X - \mathbb{E}(X)| \leqslant 13 - 10.0263$, (the condition for 7 is slightly weaker) so the probability that X is not between 7 and 13 is at most $\frac{1.5102}{(13-10.0263)^2} = 0.1708$, so the probability that the number of species she collects is between 8 and 12 inclusive is at least $1 - 0.1708 = 0.8292$.

21. *Consider the following experiment: Toss 14 fair coins. For each toss that results in a head, roll a fair (6-sided) die. Take the sum of the numbers rolled on all these dice. What is the expected outcome? [5 mins]*

    Let H be the number of heads. Let X be the outcome. We know that $H \sim B(14, \frac{1}{2})$, and that $\mathbb{E}(X|H = n) = 3.5n$. We therefore have $\mathbb{E}(X) = \mathbb{E}(\mathbb{E}(X|H)) = \mathbb{E}(3.5H) = 3.5\mathbb{E}(H) = 3.5 \times 0.5 \times 14 = 24.5$.

22. *Consider the following experiment: Roll 100 fair (6-sided) dice. For each 3 rolled, toss a fair coin, and for each 6 rolled, toss 2 fair coins. Count the number of heads on all coins tossed. What is the expected outcome? [5 mins]*

    Let X be the total number of coins tossed and let Y be the number of heads. We know that $\mathbb{E}(Y) = \mathbb{E}(\mathbb{E}(Y|X))$, and that $\mathbb{E}(Y|X = n) = \frac{n}{2}$, so $\mathbb{E}(Y) = \mathbb{E}\left(\frac{X}{2}\right) = \frac{\mathbb{E}(X)}{2}$. Let $X_i = 1$ if the ith roll is a 3, $X_i = 2$ if the ith roll is a 6 and $X_i = 0$ otherwise. Then $X = X_1 + \cdots + X_{100}$, so $\mathbb{E}(X) = \mathbb{E}(X_1) + \cdots + \mathbb{E}(X_{100}) = 100\left(\frac{1}{6}\times 1 + \frac{1}{6}\times 2\right) = 50$. Therefore $\mathbb{E}(Y) = \frac{50}{2} = 25$.

23. *Consider the following experiment: Roll a fair (6-sided) die. If the result n is odd, roll $2n$ fair dice and take the sum of the numbers. If n is even, toss $2n^2 - n$ fair coins and count the number of heads. What is the expected outcome? [5 mins]*

    If the roll is n, the expected outcome is $7n$ if n is odd and $n^2 - \frac{n}{2}$ if n is even. The overall expected outcome is therefore

    $$\frac{1}{6}(7+3+21+14+35+33) = \frac{113}{6} = 18.83333$$

24. *The point (X, Y) is uniformly distributed on the set*

    $$\{(x,y)|x+y<3, y>-1, x>-2\}$$

    .

    (a) What is the conditional expectation of X given that $Y = 2$? [5 mins]

    Given that $Y = 2$, we have $x + 2 < 3$ and $x > -2$, so we get $-2 < x < 1$, and the probability density function is constant on this set, so X is uniformly distributed on $[-2, 1]$. Its conditional expectation is therefore $\frac{1-2}{2} = -\frac{1}{2}$.

    (b) What is $\mathrm{Cov}(X, Y)$? [10 mins]

    Let A be the set $\{(x,y)|x+y<3, y>-1, x>-2\}$. Then A is a triangle with vertices at $(-2, 5)$, $(-2, -1)$ and $(4, -1)$. The area of A is therefore, 18, so $f_{X,Y}(x, y) = \frac{1}{18}$ whenever $(x, y) \in A$.

    We have that $\mathbb{E}(X) = \int\int_A \frac{x}{18} dxdy = \int_{-1}^{5}\int_{-2}^{3-y} \frac{x}{18} dxdy = \int_{-1}^{5}[\frac{x^2}{36}]_{-2}^{3-y} dy = \int_{-1}^{5} \frac{(3-y)^2-4}{36} dy = \left[\frac{(y-3)^3}{108}\right]_{-1}^{5} - \frac{2}{3} = \frac{8-(-64)}{108} - \frac{2}{3} = 0$. Similarly, we get $\mathbb{E}(Y) = 1$. Now $\mathbb{E}(XY) = \int\int_A \frac{xy}{18} dxdy = \int_{-1}^{5} \frac{y((y-3)^2-4)}{36} dy = \int_{-1}^{5} \frac{y^3-6y^2+5y}{36} dy = \left[\frac{y^4-8y^3+10y^2}{144}\right]_{-1}^{5} = \frac{625-1000+250-1-8-10}{144} = -1$. We then get $\mathrm{Cov}(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) = -1$.

25. *An insurance company finds that of its home insurance policies, only 0.5% result in a claim. Of the policies that result in a claim, the expected amount claimed is $40,000, and the standard deviation of the amount claimed is $100,000.*

    (a) What are the expected amount claimed, and the variance of the amount claimed for any policy? [If a policy does not result in a claim, the amount claimed is $0. Hint: for the variance, try to work out $\mathbb{E}(X^2)$.] [10 mins]

    Let Z be an indicator variable that is 1 if a claim is made, and 0 otherwise. We have that $\mathbb{E}(X|Z = 1) = 40000$, $P(Z = 1) = 0.005$, and $\mathbb{E}(X|Z = 0) = 0$, so $\mathbb{E}(X) = \mathbb{E}(\mathbb{E}(X|Z)) = 0.995 \times 0 + 0.005 \times 40000 = 200$.

    We are also given that $\mathrm{Var}(X|Z = 1) = 10^{10}$, so $\mathbb{E}(X^2|Z = 1) - (\mathbb{E}(X|Z = 1))^2 = 10^{10}$. That is $\mathbb{E}(X^2|Z = 1) = 40000^2 + 10^{10} = 1.16 \times 10^{10}$. Now

$\mathbb{E}(X^2) = \mathbb{E}(\mathbb{E}(X^2|Z)) = 0.995 \times 0 + 0.005 \times 1.16 \times 10^{10} = 5.8 \times 10^7$. Now we get $\text{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = 5.8 \times 10^7 - 40000 = 579960000$.

(b) If the company sets its premium to \$430, but has administrative costs of \$200 for each policy, and the company sells 500,000 policies, what is the approximate probability that the company is unable to pay out all the claims from the premiums collected? [10 mins]

The distribution of the average amount claimed per policy is approximately normal with mean \$200 and variance $\frac{579960000}{500000} = 115.992$. The company is unable to pay all the claims from the premiums if this average amount claimed is more than \$230. The probability of this occuring is $1 - \Phi\left(\frac{230-200}{\sqrt{115.996}}\right) = 1 - \Phi(2.79) = 1 - 0.9974 = 0.0026$.

26. *A car insurance company finds that of its claims, 70% are for accidents, and 30% are for thefts. The theft claims are all for \$20,000, while for the accident claims claim, the expected amount claimed is \$15,000, and the standard deviation of the amount claimed is \$30,000.*

    (a) What are the expected amount claimed, and the standard deviation of the amount claimed for any claim? [10 mins]

    Let X be the amount claimed. Let $Y = 1$ if the claim is for an accident, and 0 for a theft. We have $\mathbb{E}(X|Y = 1) = 15000$ and $\mathbb{E}(X|Y = 0) = 20000$, so $\mathbb{E}(X) = \mathbb{E}(\mathbb{E}(X|Y)) = 0.7 \times 15000 + 0.3 \times 20000 = 16500$.

    For the variance, we have $\text{Var}(X) = \mathbb{E}(\text{Var}(X|Y)) + \text{Var}(\mathbb{E}(X|Y)) = 0.7 \times 30000^2 + 0.7 \times 15000^2 + 0.3 \times 20000^2 - 16500^2 = 635250000$. The standard deviation is therefore $\sqrt{635250000} = 25204.166$.

    (b) If the company has \$16,700,000 available to cover claims, and receives 1000 claims, what is the probability that it is unable to cover the claims made? [10 mins]

    This is the probability that the average amount claimed X is more than \$16,700. By the central limit theorem, the average amount claimed can be approximated as a normal distribution with mean \$16,500 and standard deviation $\frac{25204.166}{\sqrt{1000}} = 797.03$. We let $Z = \frac{X-16500}{797.03}$, so $P(X > 16700) = P\left(Z > \frac{16700-16500}{797.03}\right) = 1 - \Phi(0.25) = 1 - 0.5987 = 0.4013$.

27. *A truncated standard normal random variable has probability density function*

$$f_X(x) = \begin{cases} \frac{e^{-\frac{x^2}{2}}}{\int_a^b e^{-\frac{x^2}{2}}} & \text{if } a \leqslant x \leqslant b \\ 0 & \text{otherwise} \end{cases}$$

That is, it is a normal distribution conditional on lying in the interval $[a, b]$.

The expectation of a truncated standard normal is given by

$$E(X) = \frac{e^{-\frac{a^2}{2}} - e^{-\frac{b^2}{2}}}{\sqrt{2\pi}(\Phi(b) - \Phi(a))}$$

An investment advisor tells you that the amount by which the market will increase next year is normally distributed with mean 10% and standard deviation 10%. The bank offers an investment product which pays the value of the market, minus a 2% fee if the market increases by between 2% and 32%. If the market rises by less than 2%, then it pays out your original investment; if the market rises by more than 32%, it pays out your original investment plus 30%. What is the expected value of this investment? [Hint: divide into the three cases where the market increases by less than 2%, the market increases by between 2% and 32%, and the market increases by over 32%.] [15 mins]

The probability that the markest increases by less than 2% is $\Phi\left(\frac{2-10}{10}\right) = \Phi(-0.8) = 1 - \Phi(0.8)$. The probability that the market increases by more than 32% is $1 - \Phi\left(\frac{32-10}{10}\right) = 1 - \Phi(2.2)$. Therefore, we can calculated the expected gain by conditioning on whether the market increases by more than 32%, 2–32% or less than 2%. Let X be the percentage by which the market increases. We then have the following:

X	Probability	Expected value
$X < 2$	$1 - \Phi(0.8)$	0
$2 \leqslant X \leqslant 32$	$\Phi(2.2) + \Phi(0.8) - 1$	$8 + 10\frac{e^{-\frac{0.8^2}{2}} - e^{-\frac{2.2^2}{2}}}{\sqrt{2\pi}(\Phi(2.2)+\Phi(0.8)-1)}$
$X > 32$	$1 - \Phi(2.2)$	30

Therefore, the expected increase of the investment is $8(\Phi(2.2) + \Phi(0.8) - 1) + 10(\frac{e^{-0.32} - e^{-2.42}}{\sqrt{2\pi}}) + 30(1 - \Phi(2.2)) = 8(0.9861 + 0.7881 - 1) + 10 \times 0.254217 + 30 \times 0.0139 = 9.15\%$.

28. *You are considering an investment. You would originally invest \$1,000, and every year, the investment will either increase by 50% with probability 0.6 or decrease by 30% with probability 0.4. You plan to use the investment after 15 years. What is the expected value of the investment after 15 years? [10 mins]*

Let X_i be the value of the investment after i years. We know that $X_0 = 1000$ and conditional on $X_i = x$, we have

$$X_{i+1} = \begin{cases} 1.5X & \text{with probability } 0.6 \\ 0.7X & \text{with probability } 0.4 \end{cases}$$

This gives $\mathbb{E}(X_{i+1}|X_i = x) = 0.6 \times 1.5 \times x + 0.4 \times 0.7x = 1.18x$. We therefore have $\mathbb{E}(X_{i+1}) = \mathbb{E}(\mathbb{E}(X_{i+1}|X_i)) = \mathbb{E}(1.18X_i) = 1.18\mathbb{E}(X_i)$. From this we deduce that $\mathbb{E}(X_i) = 1000(1.18)^i$, so $\mathbb{E}(X_{15}) = 11973.75$.

29. *If X is exponentially distributed with parameter λ_1 and Y is independent of X and normally distributed with mean μ and variance σ^2.*

(a) find the moment generating function of $X - Y$. [10 mins]

The moment generating function of X is $M_X(t) = \frac{\lambda}{\lambda - t}$. The moment generating function of Y is $M_Y(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$, so the moment generating function of $X-Y$ is $M_{X-Y}(t) = M_X(t)M_Y(-t) = \frac{\lambda e^{\frac{\sigma^2 t^2}{2}-\mu t}}{\lambda - t}$.

(b) Use the Chernoff bound with $t = \frac{-2\mu}{\sigma^2}$ to obtain a lower *bound on the probability that $X > Y$. [10 mins]*

The Chernoff bound for $t < 0$ gives $P(X - Y \leqslant 0) \leqslant e^{-0t}M_{X-Y}(t)$ Setting $t = \frac{-2\mu}{\sigma^2}$, then this gives, $P(X-Y \leqslant 0) \leqslant \frac{\lambda e^{\frac{4\mu^2}{2\sigma^2}+\frac{2\mu^2}{\sigma^2}}}{\lambda + \frac{2\mu}{\sigma^2}} = \frac{\lambda e^{\frac{4\mu^2}{\sigma^2}}}{\lambda + \frac{2\mu}{\sigma^2}}$.

Now $P(X > Y) = 1 - P(X - Y \leqslant 0) \geqslant 1 - \frac{\lambda e^{\frac{4\mu^2}{\sigma^2}}}{\lambda + \frac{2\mu}{\sigma^2}}$.

30. *Let X be exponentially distributed with parameter λ_1 and Y be independent of X and exponentially distributed with parameter λ_2, where $\lambda_1 > \lambda_2$.*

    (a) find the moment generating function of $X - Y$. [10 mins]

    The moment generating function is given by $M_{X-Y}(t) = M_x(t)M_{-Y}(t) = M_X(t)M_Y(-t) = \frac{\lambda_1}{\lambda_1 - t}\frac{\lambda_2}{\lambda_2+t}$.

    (b) Use the Chernoff bound with $t = \frac{\lambda_1-\lambda_2}{2}$ to obtain an upper *bound on the probability that $X > Y$. [10 mins]*

    The Chernoff bound states that $P(X \geqslant a) \leqslant e^{-at}M_X(t)$ for any $t > 0$, so $P(X-Y \geqslant 0) \leqslant \frac{\lambda_1\lambda_2}{(\lambda_1-t)(\lambda_2+t)}$. Substituting $t = \frac{\lambda_1-\lambda_2}{2}$, we get $P(X-Y \geqslant 0) \leqslant \frac{4\lambda_1\lambda_2}{(\lambda_1+\lambda_2)^2}$

31. *If X is exponentially distributed with parameter λ_1 and Y is exponentially distributed with parameter λ_2, where $\lambda_2 > \lambda_1$:*

    (a) find the moment generating function of $X - Y$. [10 mins]

    The moment generating function of X is $\frac{\lambda_1}{\lambda_1-t}$. The moment generating function of $-Y$ is $\frac{\lambda_2}{\lambda_2+t}$. Therefore, the moment generating function of $X-Y$ is $\frac{\lambda_1\lambda_2}{(\lambda_1-t)(\lambda_2+t)}$.

    (b) Use the Chernoff bound with $t = \frac{\lambda_1-\lambda_2}{2}$ to obtain a lower *bound on the probability that $X > Y$. [10 mins]*

    The Chernoff bound states that $P(X-Y < 0) \leqslant e^{0t}M_{X-Y}(t)$ for $t < 0$. That is, $P(X-Y < 0) \leqslant \frac{\lambda_1\lambda_2}{(\lambda_1-t)(\lambda_2+t)}$ for any $t < 0$. In particular, when $t = \frac{\lambda_1-\lambda_2}{2}$, we get $P(X-Y < 0) \leqslant \frac{\lambda_1\lambda_2}{\left(\frac{\lambda_1+\lambda_2}{2}\right)^2} = \frac{4\lambda_1\lambda_2}{(\lambda_1+\lambda_2)^2}$. We therefore get, $P(X > Y) = 1 - P(X-Y < 0) \geqslant 1 - \frac{4\lambda_1\lambda_2}{(\lambda_1+\lambda_2)^2} = \frac{(\lambda_1-\lambda_2)^2}{(\lambda_1+\lambda_2)^2}$.

32. *If X is uniformly distributed on [-1,4], and Y is uniformly distributed on the interval $[0,3]$, find the moment generating function of $X - Y$. [10 mins]*

    The moment generating function of X is $M_X(t) = \frac{e^{4t}-e^{-t}}{5t}$, and the moment generating function of $-Y$ is $M_{-Y}(t) = \frac{1-e^{-3t}}{3t}$. Therefore, the moment generating function of $X-Y$ is $\left(\frac{e^{4t}-e^{-t}}{5t}\right)\left(\frac{1-e^{-3t}}{3t}\right) = \frac{e^{4t}-e^{t}-e^{-t}+e^{-4t}}{15t^2}$.

33. *A company is planning to run an advertising campaign. It estimates that the number of customers it gains from the advertising campain will be approximately normally distributed with mean 3,000 and standard deviation 300. It also estimates that the amount spent by each customer has expected value \$50 and standard deviation \$30.*

(a) Assuming the number of customers is large enough, what is the approximate distribution of the average amount spent per customer, conditional on the number of new customers being n. [5 mins]

The average amount spent per customer is normally distributed with mean \$50 and standard deviation $\frac{30}{\sqrt{n}}$.

(b) What is the joint density function for the number of new customers and the average amount spent per customer? [10 mins]

The joint density function is given by

$$\begin{aligned} f_{X,Y}(x,y) = f_X(x)f_{Y|X}(y|X=x) &= \frac{1}{\sqrt{2\pi}300}e^{-\frac{(x-3000)^2}{2\times 300^2}}\frac{1}{\sqrt{2\pi}\frac{30}{\sqrt{x}}}e^{-\frac{(y-50)}{\left(\frac{2\times 30^2}{x}\right)}} \\ &= \frac{\sqrt{x}}{18000\pi}e^{-\frac{(x-3000)^2}{180000}-\frac{x(y-50)^2}{1800}} \end{aligned}$$

34. *A gambler is playing a slot machine in a casino. The gambler continues to bet \$1 each time. The machine has the following payouts:*

Payout	Probability
\$1	0.4
\$10	0.03
\$100	0.002
\$1000	0.00005

How many times does the gambler have to play before the probability of his having more money than he started with is less than 1%? [10 mins]

Let X_i be the amount the gambler gains on the ith play (the amount he wins minus his original \$1). If W_i is the amount he wins, then we have $\mathbb{E}(X_i) = \mathbb{E}(W_i) - 1$ and $\text{Var}(X_i) = \text{Var}(W_i)$.

Now $\mathbb{E}(W_i) = 0.4\times 1 + 0.03\times 10 + 0.002\times 100 + 0.00005\times 1000 = 0.95$ and $\mathbb{E}({X_i}^2) = 0.4\times 1 + 0.03\times 100 + 0.002\times 10000 + 0.00005\times 1000000 = 73.4$, so $\text{Var}(W_i) = 73.4 - 0.95^2 = 72.4975$. If the gambler plays n times, for large enough n, the distribution of his average gain per play A_n, is normal with mean -0.05 and variance $\frac{72.4975}{n}$. The probability that he has more money than he started with is therefore $P(A_n > 0) = 1 - \Phi\left(\frac{0.05}{\sqrt{\frac{72.4975}{n}}}\right)$. From the normal table, we have $\Phi(2.33) = 0.99$, so we need to solve

$$0.05\sqrt{\frac{n}{72.4975}} = 2.33$$

which gives $n = 72.4975 \times \frac{2.33^2}{0.05^2} = 157433$.

35. *An investor's annual profit has an expectation of $2,000 and a variance of 100,000,000. (Profits in different years are independent and identically distributed.)*

    (a) After n years, for large n, what is the approximate distribution of the investors average annual profit? [5 mins]

    The approximate distribution is normal with mean 2000 and variance $\frac{100000000}{n}$.

    (b) After how many years is the probability that the investor has made an overall loss during those years less than 0.001? [10 mins]

    The investor has made an overall loss if his average annual profit is less than 0. The probability of this is $1 - \Phi\left(\frac{2000}{\sqrt{\frac{100000000}{n}}}\right) = 1 - \Phi\left(\frac{\sqrt{n}}{5}\right)$. Now $\Phi(3.09) = 0.999$, so the probability is less than 0.001 when $\frac{\sqrt{n}}{5} > 3.09$, which happens when $\sqrt{n} > 15.45$, or $n > 238.7025$, so after 239 years.

    (c) If at any point, the investor has made a total loss of over $100,000, he must pay a fine. After how many years is the danger that he will have to pay this fine greatest, and what is the danger after this many years [assuming the approximation from (a) is valid]? [10 mins]

    He has to pay the fine after n years if the average annual loss is over $\frac{100000}{n}$. The probability of this is $1 - \Phi\left(\frac{2000+100000n^{-1}}{10000n^{-\frac{1}{2}}}\right)$. This is maximised when $\frac{2000+100000n^{-1}}{10000n^{-\frac{1}{2}}}$ is minimised. Now $\frac{2000+100000n^{-1}}{10000n^{-\frac{1}{2}}} = \frac{\sqrt{n}}{5} + \frac{10}{\sqrt{n}}$, which is minimised when $\frac{\sqrt{n}}{5} = \frac{10}{\sqrt{n}}$, or when $n = 50$. After 50 years, the risk is $1 - \Phi\left(\frac{4000\sqrt{50}}{10000}\right) = 1 - \Phi(2.828) = 1 - 0.9976 = 0.0024$.

36. *The number of visitors to a particular website on a given day is approximately normally distributed with mean 12000 and variance 2000^2. A company is considering placing an advertisement on this website. It predicts that each visitor to the website will order its product with probability 0.02, and that all visitors to the website act independently.*

    (a) What is the expectation and variance of the number of orders the company receives? [10 mins]

    Let X be the number of visitors to the website and Y the number of orders the company receives. We have that $E(Y|X = x) = 0.02x$, $E(Y^2|X = x) = 0.0004x^2 + 0.0196x$. Therefore, $E(Y) = E(E(Y|X)) = E(0.02X) = 0.02E(X) = 240$, and $E(Y^2) = E(E(Y^2|X)) = E(0.0004X^2 + 0.0196X) = 0.0004E(X^2) + 0.0196E(X) = 0.0004(12000^2 + 2000^2) + 0.0196 \times 12000 =$

$59200 + 235.2 = 59435.2$. Therefore $\mathrm{Var}(Y) = E(Y^2) - (E(Y))^2 = 59435.2 - 57600 = 1835.2$.

(b) Using a normal approximation, what is the probability that the number of orders the company receives is more than 300? [5 mins]

The probability that the company receives more than 300 orders is $1 - \Phi\left(\frac{300.5-240}{\sqrt{1835.2}}\right) = 1 - \Phi(1.41) = 1 - 0.9207 = 0.0793$.