MATH/STAT 3360, Probability
FALL 2014
Toby Kenney
Midterm Examination
Model Solutions

Each part question (a, b, c, etc.) is worth 1 mark. You should have been provided with a formula sheet and a normal distribution table. **No other notes are permitted**. Scientific calculators are permitted, but not graphical calculators.

1. *How many distict ways can the letters of the word "DISTRIBUTION" be arranged?*

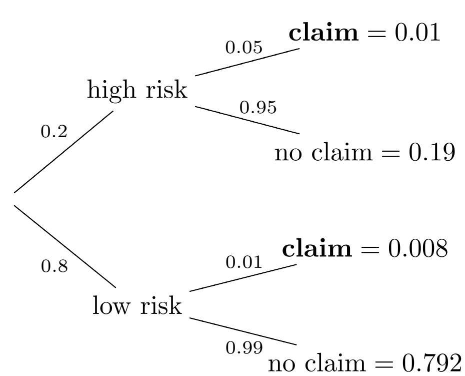
   There are a total of 12 letters, of which 3 are "I", 2 are "T", and one each for the remaining letters, "D", "S", "R", "B", "U", "O",and "N". The total number of ways is therefore $\frac{12!}{3!2!} = 11! = 39916800$.

2. *A fair coin is tossed 7 times. What is the probability that the sequence "HHTH" occurs somewhere in the 7 tosses?*

   Let A_i be the event that this sequence occurs starting in position i. The event that the sequence occurs somewhere is $A_1 \cup A_2 \cup A_3 \cup A_4$. Of the intersection of these events, only $A_1 \cap A_4$ is non-empty, so the probability is $P(A_1 \cup A_2 \cup A_3 \cup A_4) = P(A_1) + P(A_2) + P(A_3) + P(A_4) - P(A_1 \cap A_4) = \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} - \frac{1}{128} = \frac{31}{128}$.

3. *An insurance company classifies its customers as either low risk or high risk. It estimates that low risk customers have a 1% chance of making a claim each year, while high risk customers have a 5% chance of making a claim each year. 20% of customers are high risk (and the other 80%) are low risk. Given that Mr. Jones made a claim last year, what is the probability that he is a high risk customer?*

   We use a tree diagram:



   So the probability that Mr. Jones is high risk is $\frac{0.01}{0.01+0.008} = \frac{10}{18} = 0.5556$.

4. *Calculate the probability density function of X^3, where X follows a uniform distribution on the interval $[0, 1]$.*

   We first find the cumulative distribution function: $P(X^3 \leqslant x) = P(X \leqslant \sqrt[3]{x}) = \sqrt[3]{x}$ for $0 \leqslant x \leqslant 1$. This gives $f_{X^3}(x) = \frac{d}{dx}\left(x^{\frac{1}{3}}\right) = \frac{1}{3}x^{-\frac{2}{3}}$ for $0 < x < 1$, and zero outside this interval.

5. *A fair die is rolled twice. Are the following events independent?*

   (i) The first roll is 5.

   (ii) The total is 7.

   Let A be the event that the first roll is 5, and B be the event that the total is 7. We then have $P(A) = \frac{1}{6}$, $P(B) = \frac{1}{6}$ (since $B = \{16, 25, 34, 43, 52, 61\}$) and $P(A \cap B) = \frac{1}{36} = P(A)P(B)$, so A and B are independent.

6. *An insurance company pays out different levels of claims with the following probabilities:*

Claim Amount	*Probability*
\$0	*0.98*
\$3,000	*0.015*
\$40,000	*0.005*

   Find the expected value and variance of the amount claimed.

   Let C be the amount claimed. We calculate:

x	$P(C = x)$	$xP(C = x)$	$x^2P(C = x)$
\$0	0.98	0	0
\$3,000	0.015	45	135,000
\$40,000	0.005	200	8,000,000
Total		245	8,135,000

   So $\mathbb{E}(C) = 245$, and $\mathbb{E}(C^2) = 8,135,000$. This gives $\mathrm{Var}(C) = \mathbb{E}(C^2) - (\mathbb{E}(C))^2 = 8135000 - 245^2 = 8,074,975$.

7. *A Pharmaceutical company is testing the effectiveness of a new drug. It gives the drug to 20 patients. The drug passes the test if it cures at least 3 of the patients. If the probability that the drug will cure a patient is 0.2, what is the probability that it passes the test?*

   Let N be the number of patients cured. We have $N \sim B(20, 0.2)$, so $P(N \geqslant 3) = 1 - P(N = 0) - P(N = 1) - P(N = 2) = 1 - 0.8^{20} - 20 \times 0.8^{19} \times 0.2 - \binom{20}{2}0.8^{18} \times 0.2^2 = 0.794$.

8. *The number of cases of a rare disease in a given year follows a Normal distribution with mean 42 and variance 11^2. A hospital needs to prepare rooms specially for these patients, and wants to make sure that it has enough. How many rooms does it need to prepare so that there is a 95% chance that it has enough rooms for all the patients in a given year?*

Let D be the number of patients with the disease. The hospital wants to prepare r rooms, so that $P(D \leqslant r) = 0.95$, i.e., so that $\Phi\left(\frac{r-42}{11}\right) = 0.95$, which gives $\frac{r-42}{11} = 1.645$, so $r = 42 + 11 \times 1.645 = 60.095$.

9. *The time (in years) until an earthquake hits a given city follows an exponential distribution with parameter 0.04. A company builds a new building which will last either 6 years, or until the next earthquake. What is the probability that the building will be destroyed by an earthquake in the 6 years before it is demolished?*

   Let E be the time until the next earthquake. We want to find the probability that $E < 6$. This is given by $P(E < 6) = 1 - e^{-0.04\times 6} = 1 - e^{-0.24} = 0.213$.