

MATH/STAT 3460, Intermediate Statistical Theory
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In Class Examples

Question

We purchase a bag of a new type of candy, that comes in different colours. Of the first 10 sweets we take out of the bag, 3 are red, 2 are blue, 2 are green, 1 is brown, 1 is purple, and 1 is white. If we assume all colours are equally likely, what is the maximum likelihood estimate for the number of colours?

Question

You toss a coin 100 times, and get 37 heads. What is the maximum likelihood estimator for the probability of getting heads?

Question

A team of ecologists wants to know how many of a certain species of birds lives in a forest. They perform the following experiment: They capture a group of birds and mark them. They then release the birds, and capture more birds a week later, and count how many of those are marked.

Suppose the first group captured (and marked) contains 124 birds, and the second group contains 138 birds, of which 17 are marked. What is the maximum likelihood estimator for the number of birds in the forest?

Question

Let X_1, \dots, X_n be samples from a binomial distribution $B(10, p)$. What is the maximum likelihood estimator for p ?

Question

You are conducting a survey. One of the questions is potentially sensitive — the answer “YES” might be embarrassing. To avoid embarrassment, you attempt one of the following schemes:

- 1 Ask the participant to roll a die (out of sight) and if it is 6, answer “YES” regardless of the true answer.
- 2 Ask the participant to roll a die (out of sight) and if it is 6, give the opposite of the true answer.

What is the maximum likelihood estimate for the number of people who should really answer “YES” in each case?

Question

The lifetime of a light-bulb is thought to be exponentially distributed with parameter λ . 1000 light bulbs are left on for 24 hours. Within that time 8 of them break. What is the maximum likelihood estimate for λ ?

Combining Independent Events

Question

The lifetime of a light-bulb is thought to be exponentially distributed with parameter λ . 1000 light bulbs are left on for 24 hours. Within that time 8 of them break. Another set of 500 light bulbs are left on for 72 hours. Within that time 14 of them break. What is the maximum likelihood estimate for λ from the combined data?

Combining Independent Events

Question

Two people are conducting a survey with a potentially sensitive question. One of them uses technique 1 and gets 43 “YES” answers out of 200. The other uses technique 2 and gets 20 “YES” answers out of 100. What are their individual MLEs and what is the combined MLE?

Question

You toss a coin 100 times, and get 37 heads.

- (a) What is the relative likelihood that the probability of getting heads on a single roll is 0.5?
- (b) Find a 10% likelihood interval for the probability.

Question

The lifetime of a light-bulb is thought to be exponentially distributed with parameter λ . 1000 light bulbs are left on for 24 hours. Within that time 8 of them break. Find a 5% likelihood interval for λ .

Question

Let X_1, \dots, X_n be uniformly distributed on the interval $[0, a]$ for some unknown a . What is the maximum likelihood estimate for a ?

Question

Let X_1, \dots, X_n be normally distributed with mean μ and variance 1. What is the maximum likelihood estimate for μ ?

Question

Let X_1, \dots, X_n be normally distributed with mean 0 and variance σ^2 .
What is the maximum likelihood estimate for σ ?

Question

Let X_1, \dots, X_n be exponentially distributed with parameter λ . What is the maximum likelihood estimate for λ ?

Question

We take two samples from a normal distribution with unknown mean μ and variance 1. The first sample has 30 observations, and has mean 2.86. The second sample has 23 observations and has mean 3.11. What is the maximum likelihood estimate of μ from the combined data set.

Question

Let X_1, \dots, X_n be exponentially distributed with parameter λ . However, the values are censored above 1, so for any $X_i > 1$, we do not know the value of X_i , only that it is at least 1. What is the maximum likelihood estimate of λ ?

Censoring in Lifetime Experiments

Question

A company is interested in how frequently customers visit its website. When a customer visits the website, it leaves a unique cookie in the user's browser. When the user returns, it records the time since the cookie was issued. It records the following times (in days)

Days	1	2	3	4	5	6	7
Frequency Returned	10	25	31	45	53	49	51
Frequency Censored	8	15	18	30	33	34	31
Days	8	9	10	11	12	13	14
Frequency Returned	33	29	18	21	13	8	6
Frequency Censored	27	29	36	35	31	38	29

Assume that the number of days until a customer returns has a geometric distribution with probability p . What is the maximum likelihood estimate for p ?

Question

Under a certain model of evolution, for two species with phylogenetic distance t between them, the probability that a given nucleotides will be the same is $\frac{1}{4} + \frac{3}{4}e^{-\frac{4}{3}t}$. If for two given species, and a given gene, there are 532 nucleotides, of which, 346 are the same between the two species.

- What is the maximum likelihood estimate of the phylogenetic distance between them?
- Find a 10% likelihood interval for the phylogenetic distance between them.

Question

The lifetime of a light-bulb is thought to be exponentially distributed with parameter λ . 1000 light bulbs are left on for 24 hours. Within that time 8 of them break. What is the maximum likelihood estimate for the mean lifetime of a lightbulb?

Question

Suppose the number of car accidents on a typical day has a Poisson distribution with mean λ . The average time between car accidents is $\frac{1}{\lambda}$. Suppose we observe 234 car accidents over a period of 44 typical days. Calculate the MLE and a 10% likelihood interval for the average time between car accidents.

Normal Approximations

Question

We toss a coin 100 times, and get heads 37 times. Use the normal approximation to estimate a 10% likelihood interval for the probability of heads.

Question

Let X_1, \dots, X_n be independent sample from a uniform distribution on $[0, \theta]$. How large does n have to be for us to use a normal approximation to the likelihood function?

Question

Let X_1, \dots, X_n be independent samples from an exponential distribution with parameter λ . Suppose $n = 300$ and $X_1 + \dots + X_n = 824$. Find a 10% likelihood interval for λ using a normal approximation, and compare it with the true 10% likelihood interval.

Question

Let X_1, \dots, X_n be independent samples from an exponential distribution with parameter λ . Suppose $n = 300$ and $X_1 + \dots + X_n = 824$. Find a 10% likelihood interval for λ using a normal approximation to a suitable transformation of λ , and compare it with the true 10% likelihood interval.

Question

Let X_1, X_2, X_3 be independent samples whose distribution is that of a sum of two independent exponential distributions with parameters λ and 2λ . That is,

$$f_{X_i}(x) = 2\lambda(e^{-\lambda x} - e^{-2\lambda x})$$

Find the maximum likelihood estimate for λ if $X_1 = 2$, $X_2 = 3.1$ and $X_3 = 1.5$.

Question

Let X_1, \dots, X_n be independent samples from a normal distribution with mean μ and variance σ^2 , where μ and σ are unknown. What is the maximum likelihood estimate for the pair (μ, σ) ?

Two-Parameter Maximum Likelihood Estimation

Question

Let $X_1 = 4$, $X_2 = 7$, $X_3 = 5$ be independent samples from a binomial distribution, with parameters n and p . What is the maximum likelihood estimate of n and p ?

Question

We roll a die 100 times, and get 18 sixes. In another experiment, we roll the same die 100 times and get 19 fives. What is the maximum likelihood estimate for the probabilities of getting a six, and getting a five?

Two-Parameter Maximum Likelihood Estimation

Question

Let $X_i = A_i + B_i$, where A_i are uniformly distributed on $[1, a]$ and B_i are uniformly distributed on $[0, b]$. Find the maximum likelihood estimate for a and b from the X_i , if the data are $X_1 = 1.3$, $X_2 = 1.9$, $X_3 = 2.3$ and $X_4 = 3.4$. (The values of A_i and B_i are unobserved.)

Question

Let X_1, \dots, X_n be independent samples from a normal distribution with mean μ and variance σ^2 , where μ and σ are unknown. Suppose the mean of X_1, \dots, X_n is 2.3 and the sample variance $\frac{1}{n} \sum_{i=1}^n (X_i - 2.3)^2$ is 6.76.

- Find the relative likelihood function for μ and σ .
- Find a 10% likelihood region for μ and σ .

Question

Let $X_1 = 4$, $X_2 = 7$, $X_3 = 5$ be independent samples from a binomial distribution, with parameters n and p .

- (a) What is the relative likelihood function for n and p ?
- (b) Find a 1% likelihood region for n and p .

Question

Let X_1, \dots, X_n be independent samples from a normal distribution with mean μ and variance σ^2 , where μ and σ are unknown. Suppose the mean of X_1, \dots, X_n is 2.3 and the sample variance $\frac{1}{n} \sum_{i=1}^n (X_i - 2.3)^2$ is 6.76. Calculate the maximum relative likelihood function of

- (a) σ .
- (b) μ .

Question

Let $X_1 = 4$, $X_2 = 7$, $X_3 = 5$ be independent samples from a binomial distribution, with parameters n and p . What is the maximum relative likelihood function for n ?

Question

Let $X_i = A_i + B_i$, where A_i are uniformly distributed on $[1, a]$ and B_i are uniformly distributed on $[0, b]$. If the data are $X_1 = 1.3$, $X_2 = 1.9$, $X_3 = 2.3$ and $X_4 = 3.4$, find the maximum relative likelihood function for a .

Normal Approximations

Question

We roll a die 100 times, and get 18 sixes. In another experiment, we roll the same die 100 times and get 19 fives. Use the normal approximation to find a 10% likelihood region for the probabilities of rolling a six and of rolling a five.

A Dose-Response Example

Question

The probability of a response to a given dose d of drug is given by $\Phi(\alpha + \beta d)$, for some α and β .

The number of subjects receiving each dose, and the number of responses is given in the following table:

dose	0.3	0.6
number	19	20
number of responses	4	12

- Calculate the maximum likelihood estimate of α and β .
- Estimate the ED50 (That is, the dose that would produce a response rate of 50%).

A Dose-Response Example

Question

The probability of a response to a given dose d of drug is given by $1 - \frac{1}{1 + e^{\alpha + \beta d}}$, for some α and β .

The number of subjects receiving each dose, and the number of responses is given in the following table:

dose	0.1	0.3	0.4	0.5	0.6	1
number	20	19	21	23	20	19
number of responses	2	4	8	13	12	17

- Calculate the maximum likelihood estimate of α and β .
- Estimate the ED50 (That is, the dose that would produce a response rate of 50%).

An Example from Learning Theory

Question

Dog 13	00101	01111	11111	11111	11111
Dog 16	00000	00100	00001	11111	11111
Dog 17	00000	11011	00110	10111	11111
Dog 18	01100	11110	10101	11111	11111
Dog 21	00000	00011	11111	11111	11111
Dog 27	00000	01111	00101	11111	11111
Dog 29	00000	10000	00111	11111	11111
Dog 30	00000	00110	01111	11111	11111
Dog 32	00000	10101	10100	01111	10110
Dog 33	00001	00110	10111	11111	11111
Dog 34	00000	00000	11111	10111	11111
Dog 36	00000	11111	00111	11111	11111
Dog 37	00011	01001	11111	11111	11111
Dog 41	00001	01101	11111	11111	11111
Dog 42	00010	11011	11111	11111	11111

An Example from Learning Theory

Question

Dog 43	00000	00111	11111	11111	11111
Dog 45	01010	00101	11101	11111	11111
Dog 47	00001	01011	11111	11111	11111
Dog 48	01000	01000	11111	11111	11111
Dog 46	00001	10101	10101	11111	11111
Dog 49	00011	11101	11111	11111	11111
Dog 50	00101	01111	11111	10011	11111
Dog 52	00000	00111	11111	11111	11111
Dog 54	00000	00011	10100	01101	11111
Dog 57	00000	01011	11010	11111	11111
Dog 59	00101	11011	01111	11111	11111
Dog 67	00001	01111	11111	11111	11111
Dog 66	00010	10111	01011	11111	11111
Dog 69	00001	10011	10101	01011	11111
Dog 71	00001	11111	01011	11111	11111

Question

Suppose that X_1, \dots, X_n is a sample from a normal distribution with mean μ_0 and variance σ_0 . What is the distribution of the maximum likelihood estimate $(\hat{\mu}, \hat{\sigma})$ for μ and σ ?

Question

A team of ecologists wants to know how many of a certain species of birds lives in a forest. They perform the following experiment: They capture a group of birds and mark them. They then release the birds, and capture more birds a week later, and count how many of those are marked.

Suppose that they capture 124 birds in the first group, and 138 birds in the second group, and that the total number of birds in the region is 1,132. What is the distribution of the maximum likelihood estimate for the number of birds in the region?

Question

Let X_1, \dots, X_n be independent samples from a normal distribution with mean μ_0 and variance 1. What is the probability that μ_0 lies within a 10% likelihood interval?

Question

A coin has probability p of landing on heads. If we toss the coin n times, what is the probability that the true value of p lies within a 10% likelihood interval?

Chi-Square Approximation

Question

A coin has probability p of landing on heads. If we toss the coin n times, using the Chi-Square approximation, what is the probability that the true value of p lies within a 10% likelihood interval?

Chi-Square Approximation

Question

The lifetime of a light-bulb is thought to be exponentially distributed with parameter λ . 1000 light bulbs are left on for 24 hours. If the true value of the parameter is λ_0 , what is the probability that λ_0 is in a 10% likelihood interval? What do we get if we use the Chi-Squared approximation?

Confidence Intervals

Question

A team of ecologists wants to know how many of a certain species of birds lives in a forest. They perform the following experiment: They capture a group of birds and mark them. They then release the birds, and capture more birds a week later, and count how many of those are marked.

Suppose the first group captured (and marked) contains 124 birds, and the second group contains 138 birds, of which 17 are marked. Find a 95% confidence interval for the number of birds in the forest.

Question

Recall the two techniques for finding the answer to a sensitive question:

- 1 Ask the participant to roll an die (out of sight) and if it is 6, answer “YES” regardless of the true answer.
- 2 Ask the participant to roll an die (out of sight) and if it is 6, give the opposite of the true answer.

Which gives a smaller confidence interval for our final estimate of the number of people who should answer “YES”?

Question

Let X_1, \dots, X_n be samples from a normal distribution with mean μ and variance 1. Find a 95% confidence interval for μ .

Question

Let X_1, \dots, X_n be independent samples from a normal distribution with mean μ and variance σ^2 .

- (a) Find a 95% confidence region for μ and σ^2 .
- (b) Find a 95% confidence region for μ .

Results for Two-Parameter Models

Question

The probability of a response to a given dose d of drug is given by $\Phi(\alpha + \beta d)$, for some α and β .

The number of subjects receiving each dose, and the number of responses is given in the following table:

dose	0.3	0.6
number	19	20
number of responses	4	12

Use a normal approximation to find a 95% confidence interval for the ED50 (the dose that produces a response rate of 50%).

Question

A team of ecologists is performing a capture-recapture experiment to estimate the total number of birds in a forest. They have enough resources to capture a total of 500 birds in both parts of the experiment. How many birds should they capture and mark, and how many should they recapture, in order to maximise the expected information?

Question

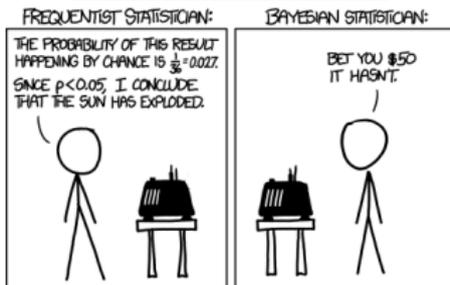
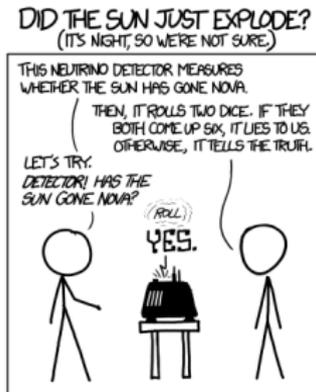
We wish to find the parameter λ from an exponential distribution, which measures the lifetimes of light bulbs. One experiment is to observe the lifetimes of n lightbulbs X_1, \dots, X_n . Another experiment is to censor after the first m have broken. That is, to observe the lifetimes until m lightbulbs have expired. What is the relative efficiency of the second experiment?

Question

A collection of light bulbs have lifetime following an exponential distribution with parameter λ . We leave 1000 light bulbs on until 10 have failed. Let $\hat{\lambda}$ be the maximum likelihood estimate of λ , based on these data.

- Calculate the bias of $\hat{\lambda}$ as an estimator for λ .
- By invariance, $\frac{1}{\hat{\lambda}}$ is the maximum likelihood estimator for the mean lifetime of a light bulb. Calculate the bias of this estimator.

Frequentists vs. Bayesians



Source: www.xkcd.com

Likelihood Ratio Tests for Simple Hypotheses

Question

A coin is tossed 100 times, and lands heads 37 times. Test the hypothesis that the coin is fair ($P(\text{Heads}) = \frac{1}{2}$).

Likelihood Ratio Tests for Simple Hypotheses

Question

A manufacturer of light bulbs claims that the mean life-time for one of its light bulbs is 5000 hours. To test this, 1000 of its light bulbs are left on for 24 hours, and during that time, 8 stop working. Test the hypothesis that the mean lifetime is 5000 hours.

Question

Let X_1, \dots, X_n be samples from a normal distribution with mean μ and variance σ^2 . Test the hypothesis that $\mu = 0$.

Question

Two companies make light bulbs. The lifetime of a light bulb from company i is thought to be exponentially distributed with parameter λ_i . Let X_1, \dots, X_n be the lifetimes of a random sample of light bulbs from company 1, and let Y_1, \dots, Y_m be the lifetimes of a random sample of light bulbs from company 2. Test the hypothesis that $\lambda_1 = \lambda_2$.

Question

Let X_1, \dots, X_k be samples from binomial distributions with parameters n_i, p_1 . Test the hypothesis that $p_1 = p_2 = \dots = p_k$.

Tests for Binomial Probabilities

Question

To test the relative effectiveness of two different drugs for an illness, each drug was given to 100 patients. Another 100 patients were given a placebo. The results are given in the following table:

	Drug A	Drug B	Placebo
Improvement	33	29	24
No improvement	67	71	76

Test the following hypotheses:

- (a) All three treatments have the same probability of improvement.
- (b) Drugs A and B have the same probability of improvement.
- (c) Assuming that A and B have the same probability of improvement, test the hypothesis that this is the same as the placebo.
- (d) Drug A has the same probability of improvement as the placebo.

Tests for Multinomial Probabilities

Question

Benford's law states that the distribution of the leading digit in a data set is given by $p(D = k) = \log_{10}(k + 1) - \log_{10}(k)$ for $k = 1, \dots, 9$. The leading digits from a number of important physical constants

http:

[//physics.nist.gov/cuu/Constants/Table/allascii.txt](http://physics.nist.gov/cuu/Constants/Table/allascii.txt)
are summarised in the table below:

Digit	1	2	3	4	5	6	7	8	9
Frequency	114	64	30	28	28	24	11	17	18

Test the hypothesis that these values follow Benford's law.

Tests for Multinomial Probabilities

Question

The novel *Sanditon* was unfinished when Jane Austen died. A fan finished the novel in an attempt to emulate the style of Jane Austen. Here are the counts of common word usage.

Word	Austen	Imitator
a	434 (433.5)	83 (83.52)
an	62 (76.3)	29 (14.73)
this	86 (84.7)	15 (16.31)
that	236 (216.3)	22 (41.79)
with	161 (171.0)	43 (33.00)
without	38 (35.2)	4 (6.85)
Total	1017	196

Test whether the relative frequencies of these words are consistent between Austen and the imitator.

Tests for Independence in Contingency Tables

Question

The admissions for a university are summarised in the following table:

	Admitted	Rejected	Total
Male	6,843	4,921	11,764
Female	3,231	2,855	6,086
Total	10,074	7,776	

Test the hypothesis that admission probability is independent of sex.

Tests for Independence in Contingency Tables

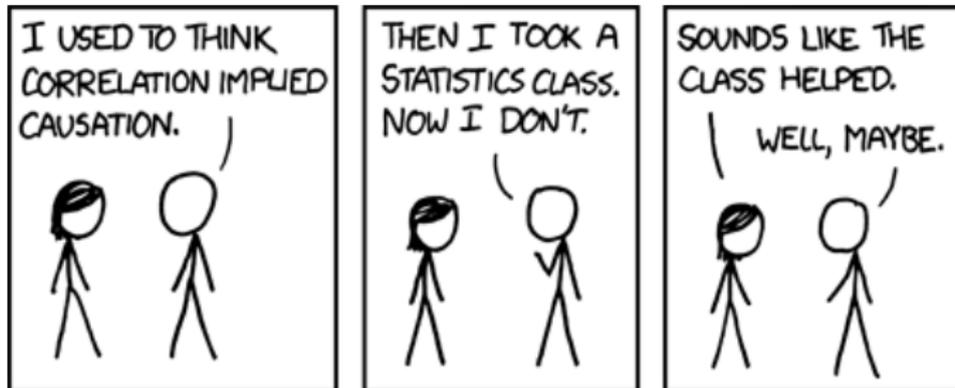
Question

A study is attempting to determine the effects of speed cameras on location. The study picks 100 locations with speed cameras, and 100 relations without speed cameras. It records whether there is an accident at each location on each of 100 days. The results are summarised in the following table:

	Speed Camera	No Speed Camera	Total
Accident	1,316	921	2,237
No Accident	8,684	9,079	17,763
Total	10,000	10,000	

Is there any association between speed cameras, and the probability of accidents?

Cause and Effect



Source: www.xkcd.com

Cause and Effect

Question

The admissions for a university are summarised in the following table:

	Subject A		Subject B		Subject C		Total
	Admit	Reject	Admit	Reject	Admit	Reject	
Male	932	1,421	2,232	2,351	3,679	1,149	11,764
Female	1,564	1,951	996	755	671	149	6,086
Total	2,498	3,372	3,228	3,106	4,350	1,298	

Test the hypothesis that admission probability for each subject is independent of sex.

Testing for Marginal Homogeneity

Question

A university is testing to ensure all its courses are the same level of difficulty. Here are the results of 153 students who took a particular two courses:

	Passed course A	Failed course A	Total
Passed course B	101	19	120
Failed course B	7	26	33
Total	108	45	

Test the hypothesis that the courses are equally difficult, i.e. the probability of a student passing each course is the same.

Testing for Marginal Homogeneity

Question

We break the results from the last question further:

		Course A			Total
		A grade	Pass	Fail	
Course B	A grade	11	8	1	20
	Pass	2	80	18	100
	Fail	0	7	26	33
Total		13	95	45	

Test the hypothesis that the courses are equally difficult, i.e. the probability of each result is the same for each course.

Significance Regions

Question

A team of ecologists wants to know how many of a certain species of birds lives in a forest. They perform the following experiment: They capture a group of birds and mark them. They then release the birds, and capture more birds a week later, and count how many of those are marked.

Suppose the first group captured (and marked) contains 124 birds, and the second group contains 138 birds, of which 17 are marked. Find a 5% significance region for the number of birds in the forest based on a likelihood ratio statistic.

Significance Regions

Question

You are conducting a survey. One of the questions is potentially sensitive — the answer “YES” might be embarrassing. To avoid embarrassment, you ask the participant to roll an die (out of sight) and if it is 6, answer “YES” regardless of the true answer. You survey 250 people, and 54 answer yes. Find a 10% significance region for the probability that the true answer for a randomly chosen person is “YES”, using the absolute difference between the observed and expected number of “YES” answers as a test statistic.

Significance Regions

Question

Let X_1, \dots, X_n be independent samples from a normal distribution with mean μ and variance 1. Find a 5% significance region for μ , using $\bar{X} - \mu$ as a test statistic. (A one-sided region).

Significance Regions

Question

Let X_1, \dots, X_{100} be independent samples from an exponential distribution. Find a 5% significance region for λ using $|\lambda - \hat{\lambda}|$ as a test statistic.

The Sufficiency Principle

Question

Let X_1, \dots, X_n be samples from an exponential distribution with parameter λ . Show that $X_1 + \dots + X_n$ is a sufficient statistic for λ .

The Sufficiency Principle

Question

Let X_1, \dots, X_n be samples from a uniform distribution on the interval $[0, a]$. Show that $\max(X_1, \dots, X_n)$ is a sufficient statistic for a .

The Sufficiency Principle

Question

Let X_1, \dots, X_n be samples from an exponential distribution with parameter λ . However, the values may be subject to censorship. Show that $X_1 + \dots + X_n$ and the number of samples M that were censored are a minimally sufficient pair of statistics for λ .

The Sufficiency Principle

Question

Let X_1, \dots, X_n be independent samples from a normal distribution with mean μ and variance σ^2 . Show that $X_1 + \dots + X_n$ and $X_1^2 + \dots + X_n^2$ are a pair of sufficient statistics for μ and σ .

Property 1

If T is sufficient for θ , then the likelihood function of θ given the observed value of T is proportional to $L(\theta; y)$.

Property 2

If T is sufficient for θ , then the conditional distribution of y given the observed value of T does not depend on θ .

Property 3

If U_1, \dots, U_k is a one-to-one transformation of T_1, \dots, T_k , then T_1, \dots, T_k is sufficient for θ if and only if U_1, \dots, U_k is.

Property 4

The maximum likelihood estimate $\hat{\theta}$ can be computed from any set of sufficient statistics.

Question

Let X_1, \dots, X_n be independent samples from an exponential distribution with parameter λ . Suppose we have $\hat{\lambda} = 0.4$. Find the exact significance level for the hypothesis $\lambda = 1$ using a likelihood ratio test, and compare it to the chi-squared approximation.

Question

Let X_1, \dots, X_n be independent samples from a normal distribution with mean μ and variance σ^2 . Calculate an exact significance level for testing the hypothesis $\mu = 0$.

Conditional Tests for Composite Hypotheses

Question

Suppose X_1, \dots, X_n are believed to be independent samples from a Poisson distribution with parameter λ , but censored at 2 (that is, values are either 0, 1, or “at least 2”). We have the following frequencies:

Value	0	1	≥ 2
Frequency	642	134	24

What is the exact significance level of a conditional likelihood ratio test for whether the censored Poisson distribution is appropriate?

Some Examples of Conditional Tests

Question

Consider the following data from a study on speed cameras:

	Speed Camera	No Speed Camera	Total
Accident	31	21	52
No Accident	69	79	148
Total	100	100	200

Perform an exact conditional test to determine the significance level of this data for testing the independence of speed cameras and accidents.