ACSC/STAT 3703, Actuarial Models I (Further Probability with Applications to Actuarial Science) WINTER 2015

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Sample Midterm Examination

This Sample examination has more questions than the actual midterm, in order to cover a wider range of questions. Estimated times are provided after each question to help your preparation.

1. *The random variable X has density function given by*

$$f(x) = \frac{15}{4}x(1-x)^2(2-x), 0 \leqslant x \leqslant 2$$

(a) calculate the hazard rate of X.

The distribution function is

$$F(x) = \frac{15}{4}\int 4x^3 - x^4 - 5x^2 + 2x dx = \frac{15}{4}\left(x^4 - \frac{x^5}{5} - \frac{5}{3}x^3 + x^2\right) = \frac{x^2}{4}\left(15x^2 - 3x^3 + 15 - 25x\right)$$

The survival function is therefore

$$\begin{aligned} S(x) &= \frac{1}{4}\left(4 + 3x^5 + 25x^3 - 15x^4 - 15x^2\right) = \frac{1}{4}(x-2)\left(3x^4 - 9x^3 + 7x^2 - x - 2\right) \\ &= \frac{1}{4}(x-2)^2(3x^3 - 3x^2 + x + 1) \end{aligned}$$

The hazard rate is therefore

$$\lambda(x) = \frac{f(x)}{S(x)} = \frac{\frac{15}{4}x(1-x)^2(2-x)}{\frac{1}{4}(x-2)^2(3x^3 - 3x^2 + x + 1)} = \frac{15x(1-x)^2}{(x-2)(3x^3 - 3x^2 + x + 1)}$$

(b) Calculate the kurtosis of X

The raw moments are

$$\mu = 3.75\int_0^2 x^2(x-1)^2(x-2)dx = 3.75\int_0^2 (4x^4+2x^2-x^5-5x^3)dx$$
$$= 3.75\left(0.8\times 2^5 + \frac{2}{3}\times 2^3 - \frac{2^6}{6} - 1.25\times 2^4\right) = 96+20-40-75 = 1$$
$$\mu_2' = 3.75\left(\frac{4}{6}\times 2^6 + \frac{2}{4}\times 2^4 - \frac{2^7}{7} - 2^5\right) = 160+30-\frac{480}{7}-120 = \frac{10}{7}$$
$$\mu_3' = 3.75\left(\frac{4}{7}\times 2^7 + \frac{2}{5}\times 2^5 - \frac{2^8}{8} - \frac{5}{6}\times 2^6\right) = \frac{1920}{7}+48-120-200 = \frac{16}{7}$$
$$\mu_4' = 3.75\left(\frac{4}{8}\times 2^8 + \frac{2}{6}\times 2^6 - \frac{2^9}{9} - \frac{5}{7}\times 2^7\right) = 480+80-\frac{640}{3}-\frac{2400}{7} = \frac{80}{21}$$

The centralised moments are

$$\mu_2 = \frac{10}{7} - 1^2 = \frac{3}{7}$$
$$\mu_3 = \frac{16}{7} - 3\times\frac{10}{7}\times 1 + 2\times 1^3 = 0$$
$$\mu_4 = \frac{80}{21} - 4\times\frac{16}{7}\times 1 + 6\times\frac{10}{7}\times 1^2 - 3\times 1^4 = \frac{5}{21}$$

The Kurtosis is therefore

$$\frac{\left(\frac{5}{21}\right)}{\left(\frac{3}{7}\right)^2} = \frac{35}{27}$$

2. *Losses follow a Pareto distribution with $\alpha = 3$. How large can θ be if the insurance company wants to limit its Value at Risk at the 95% level to \$15,000?*

   Recall that Value at Risk is the 95th percentile, so we need to solve

$$\left(\frac{\theta}{15000+\theta}\right)^3 = 0.05$$
$$\frac{\theta}{15000+\theta} = 0.05^{\frac{1}{3}}$$
$$\frac{15000+\theta}{\theta} = 20^{\frac{1}{3}}$$
$$\frac{15000}{\theta} = 20^{\frac{1}{3}} - 1$$
$$\theta = \frac{15000}{20^{\frac{1}{3}}-1} = 8749.33$$

3. *Calculate the moment generating function of a sum of 5 independent beta random variables with parameters 3 and 2*

$$\begin{aligned}\mathbb{E}(e^{tX}) &= 12\int_0^1 x^2(1-x)e^{tx}dx = 12\left(\left[\frac{x^2(1-x)e^{tx}}{t}\right]_0^1 - \int_0^1 \frac{(2x-3x^2)e^{tx}}{t}dx\right)\\ &= 12\left(-\left[\frac{(2x-3x^2)e^{tx}}{t^2}\right]_0^1 + \int_0^1 \frac{(2-6x)e^{tx}}{t^2}dx\right)\\ &= 12\left(\frac{e^t}{t^2} + \left[\frac{(2-6x)e^{tx}}{t^3}\right]_0^1 + \int_0^1 \frac{6e^{tx}}{t^3}dx\right) = 12\left(\frac{e^t}{t^2} - \frac{2}{t^3} - \frac{4e^t}{t^3} + \frac{6e^t}{t^4} - \frac{6}{t^4}\right)\\ &= \frac{12(t^2-4t+6)e^t - 24(t+3)}{t^4}\end{aligned}$$

The moment generating function of a sum of 5 beta random variables is therefore

$$\left(\frac{12(t^2-4t+6)e^t - 24(t+3)}{t^4}\right)^5$$

4. *Which distribution has a heavier tail: a gamma distribution with $\alpha = 4$ and $\theta = 400$, or a Weibull distribution with $\tau = 4$ and $\theta = 400$? [Use any reasonable method for comparing tail-weight.]*

The easiest method is to look at the ratio of their density functions. This is

$$\frac{x^3e^{-\frac{x}{400}} \times x}{6\times 4\left(\frac{x}{400}\right)^4 e^{-\left(\frac{x}{400}\right)^4}} = \frac{e^{\left(\frac{x}{400}\right)^4 - \frac{x}{400}}}{24}$$

This clearly tends to ∞ as $x \to \infty$, so the gamma distribution has a heavier tail.

Alternative solution:

The nth moment of the gamma distribution is $\mu_n' = 400^n\frac{\Gamma(n+4)}{\Gamma(4)}$, while the nth moment of the Weibull distribution is $400^n\Gamma\left(1+\frac{n}{4}\right)$. As n gets large, the moments of the gamma distribution are larger than the moments of the Weibull distribution, so the gamma has the heavier tail.

[Other approaches include comparing hazard rates, or survival functions, but these are harder to compute for the gamma distribution.]

5. *Recall that desirable coherence properties for measures of risk are:*

   - *Subadditivity*
   - *Monotonicity*

- *Positive homogeneity*
- *Translation invariance*

Which properties are satisfied by the risk measure given by the measure $r(X) = \mu + \pi_{0.9}$ *(the mean plus the 90th percentile)?*

The mean satisfies all of the properties, while the 90th percentile satisfies all except subadditivity, so the sum of risk measures must also satisfy this property. The only thing remaining is to check whether subadditivity is satisfied. Let X be a Bernoulli random variable with $p = 0.09$. The 90th percentile of X is therefore 0, and the mean is 0.09. Let Y be another Bernoulli variable mutually exclusive with X (That is, we can't have $X = Y = 1$) with $p = 0.02$. Then $X + Y$ is a Bernoulli random variable with $p = 0.11$, so we have $r(X + Y) = 0.11 + 1 = 1.11$, while $r(X) = 0.09$ and $r(Y) = 0.02$, so the measure is not subadditive.

6. *Calculate the TVaR of a gamma distribution with* $\alpha = 3$ *and* $\theta = 2000$ *at the* 0.99 *level. [The VaR at the 0.99 level is 16,811.894]*

   This is the conditional expectation given that the value is above the 99th percentile. Let π be the 99th percentile of a gamma distribution with $\alpha = 3$ and $\theta = 1$. Then the TVaR is $2000\frac{\int_\pi^\infty x^3e^{-x}\,dx}{\int_\pi^\infty x^2e^{-x}\,dx}$. Integrating by parts, we get

   $$\int_\pi^\infty x^3e^{-x}dx = [-x^3e^{-x}]_\pi^\infty + \int_\pi^\infty 3x^2e^{-x}dx$$

   and $\int_\pi^\infty x^3e^{-x}dx = [-x^2e^{-x}]_\pi^\infty + \int_\pi^\infty 2xe^{-x}dx = \pi^2e^{-\pi} + 2\pi e^{-\pi} + 2e^{-\pi}$. The TVaR is therefore $2000\left(\frac{\pi^3+3\pi^2+6\pi+6}{\pi^2+2\pi+2}\right)$ Furthermore, π is the 99th percentile, meaning that $\int_\pi^\infty x^2e^{-x}dx = 0.01\Gamma(3) = 0.02$. We therefore have that $(\pi^2 + 2\pi + 2)e^{-\pi} = 0.02$ and the TVaR is

   $$2000\left(\frac{\pi^3}{0.02e^\pi} + 3\right)$$

   We are given that $\pi = \frac{16811.894}{2000} = 8.405947$, so the TVaR is

   $$2000\left(\frac{8.405947^3}{0.02e^{8.405947}} + 3\right) = \$19,277.11$$

7. *Claims follow a Pareto distribution with* $\alpha = 4$. *There is a policy limit which is currently exceeded by 0.16% of claims. There is uniform inflation of 8% per year on claim amounts. What proportion of claims will exceed the policy limit in 4 years time? [The policy limit does not change in these 4 years.]*

   In 4 years time, the loss distribution will be Pareto with $\alpha = 4$ and $\theta = (1.08)^4\theta_0$ (where θ_0 is the current parameter value). We have that the policy limit l is set so that $\left(\frac{\theta_0}{\theta_0+l}\right)^4 = 0.0016$. This gives $\frac{\theta_0+l}{\theta_0} = 5$, so $l = 4\theta_0$. In 4 years time, we will have $\theta = (1.08)^4\theta_0$, so the probability of exceeding the policy limit will be $\left(\frac{(1.08)^4\theta_0}{(1.08)^4\theta_0+4\theta_0}\right)^4 = \left(\frac{(1.08)^4}{(1.08)^4+4}\right)^4 =$ 0.004149182, so 0.415% of claims will exceed the limit.

8. *An insurance company deals with three types of claim:*

   - *10% of claims are for fire damage. These claims follow a Pareto distribution with $\alpha = 3$ and $\theta = 300,000$.*
   - *60% of claims are for weather damage. These claims follow a Weibull distribution with $\tau = \frac{1}{3}$ and $\theta = 10,800$.*
   - *The remaining 30% of claims are for break-ins. These claims follow a Weibull distribution with $\tau = 3$ and $\theta = 1000$.*

   Calculate the probability that a randomly chosen claim exceeds $400,000.

   For a fire damage claim, the probability of exceeding $400,000 is $\left(\frac{300000}{300000+400000}\right)^3 = \frac{27}{343}$. For a weather damage claim, the probability of exceeding $400,000 is $e^{-\left(\frac{400000}{10800}\right)^{\frac{1}{3}}} = e^{-\frac{10}{3}}$. For a break-in claim, the probability of exceeding $400,000 is $e^{-\left(\frac{400000}{1000}\right)^3} = e^{-64000000}$. The overall probability of a random claim exceeding $400,000 is therefore

$$0.1 \times \frac{27}{343} + 0.6e^{-\frac{10}{3}} + 0.3e^{-64000000} = 0.02927612$$

9. *You observe the following sample of insurance losses:*

   1.6 3.6 3.8 4.2 5.6

   Using a Kernel density model with Gaussian (normal) kernel with standard deviation 1.2, estimate the probability that a loss exceeds 5.5.

   This probability is given by

$$1 - \frac{1}{5}\left(\Phi\left(\frac{5.5-1.6}{1.2}\right) + \Phi\left(\frac{5.5-3.6}{1.2}\right) + \Phi\left(\frac{5.5-3.8}{1.2}\right) + \Phi\left(\frac{5.5-4.2}{1.2}\right) + \Phi\left(\frac{5.5-5.6}{1.2}\right)\right)$$
$$= 1 - \frac{1}{5}\left(\Phi(3.25) + \Phi(1.58) + \Phi(1.42) + \Phi(1.08) + \Phi(-0.08)\right)$$
$$= 1 - \frac{1}{5}(0.9994 + 0.9429 + 0.9222 + 0.8599 + 0.4681) = 1 - \frac{4.1925}{5} = 1 - 0.8385 = 0.1615$$

10. *You observe the following sample of insurance losses:*

    1.6 3.6 3.8 4.2 5.6

Using a Kernel density model with triangular kernel with bandwidth 2, estimate the probability that a loss exceeds 5.5. A triangular kernel with bandwidth b centred at x_0 is given by the density function

$$f(x) = \begin{cases} \frac{x+b-x_0}{b^2} & \text{if } x_0 - b < x < x_0 \\ \frac{x_0+b-x}{b^2} & \text{if } x_0 < x < x_0 + b \\ 0 & \text{otherwise} \end{cases}$$

The probability of exceeding 5.5 is given by

$$\frac{1}{5}(0 + 0.005 + 0.045 + 0.245 + 0.545) = \frac{0.84}{5} = 0.168$$

11. *An insurance company models its investment gains over a period of t years as e^X where X follows a gamma distribution with parameters $\alpha = 2t$ and $\theta = 0.05$. Calculate the density function for its investment gains over a 4-year period.*

We are calculating the density function of e^X, where X follows a gamma distribution with $\alpha = 8$ and $\theta = 0.05$. The density of this is

$$f_{e^X}(a) = \frac{(\frac{\log(a)}{0.05})^8 e^{-\frac{\log(a)}{0.05}}}{a\log(a)\Gamma(8)} = \frac{20^8 \log(a)^7 a^{-21}}{7!}$$

12. *Given $\Theta = \theta$, the lifetime of a computer follows an inverse exponential distribution with parameter θ. Θ follows a gamma distribution with $\alpha = 2$, and $\theta = 3$. What is the distribution of the lifetime of a randomly chosen computer?*

The lifetime of a randomly chosen computer has density function:

$$\begin{aligned} f(x) &= \int_0^\infty \frac{\theta}{x^2} e^{-\left(\frac{\theta}{x}\right)} \frac{\theta}{9} e^{-\frac{\theta}{3}} d\theta = \frac{1}{9x^2} \int_0^\infty \theta^2 e^{-\theta\left(\frac{1}{x}+\frac{1}{3}\right)} d\theta \\ &= \frac{1}{9x^2}\left(\left[-\theta^2 \frac{3x}{x+3} e^{-\theta\frac{x+3}{3x}}\right]_0^\infty + \frac{3x}{x+3}\int_0^\infty 2\theta e^{-\theta\frac{x+3}{3x}} d\theta\right) \\ &= \frac{1}{9x^2}\left(\left[-2\theta\left(\frac{3x}{x+3}\right)^2 e^{-\theta\frac{x+3}{3x}}\right]_0^\infty + 2\left(\frac{3x}{x+3}\right)^2 \int_0^\infty e^{-\theta\frac{x+3}{3x}} d\theta\right) \\ &= \frac{2}{9x^2}\left(\frac{3x}{(x+3)}\right)^3 = \frac{6x}{(x+3)^3} \end{aligned}$$

So the lifetime follows a Pareto distribution with $\alpha = 1$ and $\theta = 3$.

13. *The mortality rate of a man aged x is modelled as being $\lambda e^{0.1x}$, where λ follows a gamma distribution with $\alpha = 3$ and $\theta = 0.001$. Calculate the probability of a man aged 40 surviving to age 90.*

given λ, the probability of surviving to age t is $e^{-\int_0^t \lambda e^{0.1x}dx}$. The probability of surviving to age t is therefore

$$\int_0^\infty \frac{1000^3}{2}\lambda^2 e^{-1000\lambda}e^{-\lambda\int_0^t e^{0.1x}dx}d\lambda = \frac{1000^3}{2}\int_0^\infty \lambda^2 e^{-\lambda\left(1000+\int_0^t e^{0.1x}dx\right)}d\lambda$$

$$= \frac{1000^3}{\left(1000+\int_0^t e^{0.1x}dx\right)^3} = \frac{1000^3}{(1000+10(e^{0.1t}-1))^3}$$

That is, $e^{0.1t}$ follows a Pareto distribution with $\theta = 1000$, $\alpha = 3$. The probability of surviving to age 90 is therefore $\frac{1000^3}{(1000+10(e^9-1))^3} = 0.001330045$; the probability of surviving to age 40 is $\frac{1000^3}{(1000+10(e^4-1))^3} = 0.8773083$. The probability of an individual aged 40 surviving to 90 is therefore $\frac{\left(1000+10(e^9-1)\right)^3}{(1000+10(e^4-1))^3} = 0.001516052$.

14. *Recall that the limit of a transformed beta distribution as $\tau \to \infty$, $\theta \to 0$ and $\theta\tau^{\frac{1}{\gamma}} \to \xi$ is an inverse gamma with $\theta = \xi$ and $\alpha = \alpha$. What is the limit of an inverse Pareto distribution as $\tau \to \infty$ and $\theta \to 0$ with $\tau\theta = \xi$.*

The inverse Pareto distribution is the special case of the transformed beta distribution with $\alpha = \gamma = 1$. Its limit is therefore the special case of the inverse transformed gamma with $\alpha = 1$ and $\gamma = 1$. This is the inverse exponential distribution.

15. *Let X have density function given by*

$$f(x) = \left(\frac{\theta}{1+\theta}\right)^x \log\left(\frac{1+\theta}{\theta}\right)$$

for $0 < x$.

(a) Show that the distribution of X is from the linear exponential family, and calculate the functions $p(x)$, $q(\theta)$, and $r(\theta)$.

We rewrite the density function of X as

$$f(x) = \log\left(\frac{1+\theta}{\theta}\right)e^{x\log\left(\frac{\theta}{1+\theta}\right)}$$

We therefore get $p(x) = 1$, $q(\theta) = \log\left(\frac{\theta}{1+\theta}\right)$ and $r(\theta) = \log\left(\frac{\theta}{1+\theta}\right)$.

[This is an exponential distribution with hazard rate $\log\left(\frac{1+\theta}{\theta}\right)$.]

(b) Calculate the variance of X as a function of θ.

We have that

$$\mu = \frac{q'(\theta)}{q(\theta)r'(\theta)} = \frac{1}{q(\theta)} = \frac{1}{\log\left(\frac{1+\theta}{\theta}\right)}$$

and

$$\mu_2 = \frac{\mu'(\theta)}{r'(\theta)} = \frac{1}{\log\left(\frac{1+\theta}{\theta}\right)^2}$$

16. *The number of claims experienced by an insurance company in a given year follow a Poisson distribution with mean 30. Of these claims, 10% are for fires and 15% are for floods. What is the probability that in a given year the company experiences exactly 2 claims for fires and at most 2 claims for floods?*

    The number of claims for fires follows a Poisson distribution with mean 3; the number of claims for floods follows a Poisson distribution with mean 4.5; and these two numbers are independent. Therefore, the probability that the company receives exactly two claims for fires and at most 2 claims for floods is $e^{-3}\frac{3^2}{2!}e^{-4.5}\left(1+4.5+\frac{4.5^2}{2!}\right) = 15.625 \times 4.5e^{-7.5} = 70.3125e^{-7.5} = 0.03888874$.

17. *For a driver with safety rating $\Theta = \theta$, the number of claims made in a year follows a Poisson distribution with parameter θ. For a random driver, Θ follows a gamma distribution with parameter $\alpha = 4$ and $\theta = 0.2$. What is the probability that a randomly chosen driver makes no claims in a given year?*

    The number of claims made by a random driver is a negative binomial with $r = \alpha = 4$ and $\beta = \theta = 0.2$. The probability of a random driver making no claims is therefore $\left(\frac{1}{1+0.2}\right)^4 = 0.4822531$.

    [We can derive this directly from the mixture: Given $\Theta = \theta$, the probability of making no claims is $e^{-\theta}$, so the overall probability of making no claims is

    $$\int_0^\infty \frac{\theta^4 e^{-\frac{\theta}{0.2}}e^{-\theta}}{0.2^4\Gamma(4)}d\theta = \int_0^\infty \frac{\theta^4 e^{-6\theta}}{0.2^4\Gamma(4)}d\theta = \frac{1}{1.2^4} = 0.4822531$$

    which is the same as we calculated above.]

18. *An insurance company models the number of claims received with a distribution from the $(a, b, 1)$-class. It calculates that the probability of receiving exactly 5 claims is 0.1; the probability of receiving exactly 6 claims is 0.04; the probability of receiving exactly 8 claims is 0.0027. What is the modified probability of receiving no claims?*

We have $p_5 = 0.1$, $p_6 = 0.04$ and $p_8 = 0.0027$. This gives

$$\begin{aligned} a + \frac{b}{6} &= \frac{p_6}{p_5} = 0.4 \\ \left(a + \frac{b}{7}\right)\left(a + \frac{b}{8}\right) &= \frac{p_8}{p_6} = \frac{27}{400} \end{aligned}$$

$$\begin{aligned} 6a + b &= 2.4 \\ (7a + b)(8a + b) &= 3.78 \end{aligned}$$

Substituting the first equation into the second gives

$$\begin{aligned} (a + 2.4)(2a + 2.4) &= 3.78 \\ (a + 2.4)(a + 1.2) &= 1.89 \\ (a + 1.8)^2 - 0.36 &= 1.89 \\ a + 1.8 &= \pm\sqrt{2.25} = \pm 1.5 \end{aligned}$$

This gives two solutions $a = -0.3$ and $a = -3.3$. The first solution gives $b = 4.2$, which is a binomial distribution with $n = 15$ and $p = \frac{3}{13}$. The second solution gives $b = 22.2$, which is not divisible by a, so this does not give a valid distribution. Therefore, the solution is a zero-modified binomial. Under the binomial distribution we would have $p_5 = \binom{15}{5}\left(\frac{3}{13}\right)^5\left(\frac{10}{13}\right)^{10} = 0.1425645$, so this probability has been multiplied by $\frac{1}{1.425645} = 0.701437$. For the original distribution, the probability $p_0 = \left(\frac{3}{13}\right)^{15} = 2.803293e{-10}$. The new probability is $p_0 = 1 - 0.701437\left(1 - \left(\frac{3}{13}\right)^{15}\right) = 0.298563$.

19. *An insurance company observes the following claim numbers on a group insurance policy:*

Number of Claims	*Frequency*
0	*12,345*
1	*4,521*
2	*874*
3	*130*
4	*17*
5	*2*
6 or more	*0*

By calculating $k\frac{p_k}{p_{k-1}}$*, decide which distributions from the* $(a,b,0)$*-class are most appropriate.*

The values are

k	$k\frac{p_k}{p_{k-1}}$
1	0.3662211
2	0.3866401
3	0.4462243
4	0.5230769
5	0.5882353

We see that these numbers are increasing, which suggests we should choose $a \geqslant 0$, so a negative binomial or a Poisson distribution might be appropriate.

20. *An insurance company models loss frequency for an individual as following a zero-modified logarithmic distribution with* $p_0 = 0.8$ *and* $\beta = 3$*. What is the probability that this individual experiences at least 3 losses?*

    We have $r = 0$ and $\beta = 3$. This gives $a = 0.75$ and $b = -0.75$. We also have $p_1^T = \frac{3}{4\log 4} = 0.5410106$. The zero-modified probability is therefore $p_1 = 0.2 \times 0.5410106 = 0.1082021$. We therefore get $p_2 = (0.75 - 0.375)p_1 = 0.375 \times 0.1082021 = 0.0405758$. This gives that the probability of at least 3 losses is $1 - 0.8 - 0.1082021 - 0.0405758 = 0.05122207$.

21. *Losses follow a compond Poisson-Poisson distribution with parameters 2 and 4. Calculate the probability that there are more than 2 losses.*

    The probability that there are no losses is

$$e^{-2}\sum_{n=0}^{\infty}\frac{2^n e^{-4n}}{n!} = e^{2(e^{-4}-1)} = 0.1403847$$

    We have $a = 0, b = 2$, so the recurrence gives

$$p_n = e^{-2}\sum_{k=1}^{n}\frac{2^k \times 2k}{n \times k!}p_{n-k}$$

    so we get

$$p_1 = 4e^{-2} \times 0.1403847 = 0.07599601$$
$$p_2 = 2e^{-2} \times 0.07599601 + 4e^{-2} \times 0.1403847 = 0.09656589$$

22. *The number of claims on policies from one group with 200 members follows a compound Poisson-Geometric distribution with parameters* $\lambda = 3$ *and* $\beta = 2$*. The number of claims on policies from another group with 250 members follows a compound Poisson-Poisson distribution with parameters 2 and 1. Next year, the first group is increasing to 400 members and*

the second group is increasing to 400 members. Calculate the probability that there are at least 3 claims next year.

Next year, the number of claims from the first group will follow a compound Poisson-Geometric distribution with $\lambda = 6$ and $\beta = 2$. The number of claims from the second group will follow a compound Poisson-Poisson distribution with parameters 3.2 and 1.6. The sum of these distributions is a compound Poisson distribution with $\lambda = 9.2$ and secondary distribution a mixture of a Geometric distribution and a Poisson distribution. The probabilities of this secondary distribution are:

n	$P(X = n)$
0	0.3453494
1	0.2728856
2	0.1605974

The primary distribution is from the $(a, b, 0)$-class with $a = 0$, $b = 9.2$, so the recurrence is

$$q_n = \sum_{k=1}^{n} \frac{9.2k}{n} p_k q_{n-k}$$

We can calculate $q_0 = e^{-9.2} \sum_{n=0}^{\infty} \frac{9.2^n \times 0.3453494^n}{n!} = e^{-9.2(1-0.3453494)} = 0.00242291$

We then use the recurrence to get

$$p_0 = 0.00242291$$
$$p_1 = 0.007698106$$
$$p_2 = 0.01580911$$

Therefore the probability that the company receives at least 3 claims is $1 - 0.00242291 - 0.007698106 - 0.01580911 = 0.9741$.

23. *The number of claims follows a mixture distribution. Given $\Theta = \theta$, the number of claims follows a negative binomial distribution with $r = 4$ and $\beta = \frac{\theta}{4}$. Θ follows a Pareto distribution with $\theta = 4$ and $\alpha = 3$. What is the probability that the number of claims is exactly 3?*

Given $\Theta = \theta$, the probability that the number of claims is exactly 3 is

$$\binom{6}{3} \left(\frac{\theta}{4+\theta}\right)^3 \left(\frac{4}{4+\theta}\right)^4$$

The overall probability that the number of claims is 3 is therefore

$$\int_0^\infty 20\left(\frac{\theta}{4+\theta}\right)^3\left(\frac{4}{4+\theta}\right)^4\frac{3\times 4^3}{(4+\theta)^4}d\theta = 60\times 4^7\int_0^\infty \frac{\theta^3}{(4+\theta)^{11}}d\theta$$

$$= 60\times 4^7\int_4^\infty \frac{(u-4)^3}{u^{11}}du = -60\times 4^7\left[\frac{u^{-7}}{7}-\frac{12u^{-8}}{8}+\frac{48u^{-9}}{9}-\frac{64u^{-10}}{10}\right]_4^\infty$$

$$= 60\times 4^7\left(\frac{1}{7\times 4^7}-\frac{12}{8\times 4^8}+\frac{48}{9\times 4^9}-\frac{64}{10\times 4^{10}}\right)$$

$$= \frac{60\times 4^7}{4^7}\left(\frac{1}{7}-\frac{3}{8}+\frac{3}{9}-\frac{1}{10}\right) = 60\frac{120-315+280-84}{840} = \frac{1}{14}$$