ACSC/STAT 3703, Actuarial Models I (Further Probability with Applications to Actuarial Science) Winter 2015

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Homework Sheet 1
Model Solutions

Basic Questions

1. *The survival function for an inverse Weibull distribution is given by* $S(x) = 1 - e^{\left(\frac{\theta}{x}\right)^\tau}$*. Calculate the hazard-rate.*

   The density function is given by $f(x) = -\frac{d}{dx}S(x) = \frac{\tau\theta^\tau}{x^{\tau+1}}e^{\left(\frac{\theta}{x}\right)^\tau}$. The hazard rate function is given by

$$\lambda(x) = \frac{\frac{\tau\theta^\tau}{x^{\tau+1}}e^{\left(\frac{\theta}{x}\right)^\tau}}{1 - e^{\left(\frac{\theta}{x}\right)^\tau}}$$

2. *A continuous random variable has moment generating function given by* $M(t) = (1-4t)^{-2}e^{1-\sqrt{1-2t}}$ *for* $t < \frac{1}{2}$*. Calculate its coefficient of variation.*

   We can calculate the moments by differentiating the moment generating function at 0. We have

$$M'(t) = \left((1-4t)^{-2}(1-2t)^{-\frac{1}{2}} + 8(1-4t)^{-3}\right)e^{1-\sqrt{1-2t}}$$
$$M''(t) = \Big(\left((1-4t)^{-2}(1-2t)^{-\frac{1}{2}} + 8(1-4t)^{-3}\right)(1-2t)^{-\frac{1}{2}} + 8(1-4t)^{-3}(1-2t)^{-\frac{1}{2}} + (1-4t)^{-2}(1-2t)^{-\frac{3}{2}} + 96(1-4t)^{-4}\Big)e^{1-\sqrt{1-2t}}$$

   We get $M'(0) = 9$ and $M''(0) = 114$. This gives the variance is $114 - 9^2 = 33$, so the coefficient of variation is $\frac{\sqrt{33}}{9} = 0.6382847385$.

3. *Calculate the mean excess loss function for a distribution with survival function given by* $S(x) = \left(1 - \frac{x}{130}\right)^{\frac{1}{5}}$*.*

   The mean excess loss function at d is given by $\frac{\int_d^{130} S(x)dx}{S(d)} = \frac{\int_d^{130}\left(1-\frac{x}{130}\right)^{\frac{1}{5}}dx}{S(d)}$. We substitute $t = 1 - \frac{x}{130}$, so we get the mean excess loss is given by $\frac{\int_0^{1-\frac{d}{130}} 130t^{\frac{1}{5}}\,dt}{\left(1-\frac{x}{130}\right)^{\frac{1}{5}}} = \frac{650\left(1-\frac{d}{130}\right)^{\frac{6}{5}}}{6\left(1-\frac{d}{130}\right)} = 108.333333\left(1 - \frac{d}{130}\right)$.

4. *Find the equilibrium distribution for a Weibull distribution with survival function given by* $S(x) = e^{-\left(\frac{x}{\theta}\right)^\tau}$.

   We have that $\mathbb{E}(X) = \theta\Gamma\left(1 + \frac{1}{\tau}\right)$, so the density of the equilibrium distribution is

$$f_e(x) = \frac{e^{-\left(\frac{x}{\theta}\right)^\tau}}{\theta\Gamma\left(1 + \frac{1}{\tau}\right)}$$

Standard Questions

5. *A Burr distribution has survival function*

$$S(x) = \left(\frac{1}{1 + \left(\frac{x}{\theta}\right)^\gamma}\right)^\alpha$$

   Consider the two Burr distributions $\alpha = 2, \gamma = 3, \theta = 20$ *and* $\alpha = 3, \gamma = 2, \theta = 40$. *Which has the heavier tail when measured by the hazard rate function?*

   The density function is given by differentiating the survival function and multiplying by -1.

$$f(x) = \alpha\gamma\frac{x^{\gamma-1}}{\theta^\gamma}\left(\frac{1}{1 + \left(\frac{x}{\theta}\right)^\gamma}\right)^{\alpha+1}$$

   The hazard rate is given by dividing this by the survival function. That is

$$\lambda(x) = \frac{\alpha\gamma\frac{x^{\gamma-1}}{\theta^\gamma}\left(\frac{1}{1+\left(\frac{x}{\theta}\right)^\gamma}\right)^{\alpha+1}}{\left(\frac{1}{1+\left(\frac{x}{\theta}\right)^\gamma}\right)^\alpha} = \frac{\alpha\gamma x^{\gamma-1}}{\theta^\gamma + x^\gamma}$$

   The hazard rate functions for the two distributions are therefore:

$$\frac{6x^2}{20^3 + x^3} \quad \text{and} \quad \frac{6x}{40^2 + x^2}$$

   The ratio of the hazard rates is therefore

$$\frac{40^2 x + x^3}{20^3 + x^3}$$

   For large x, this is greater than 1, but converges to 1 as $x \to \infty$, so the tails are similar, but the second distribution has a slightly heavier tail.

6. *An insurance company is trying to fit a paralogistic distribution to its claims data. The survival function for this distribution is given by*

$$S(x) = \left(\frac{1}{1 + \left(\frac{x}{\theta}\right)^\alpha}\right)^\alpha$$

It is very important for the insurance company to correctly model the expected value and the 95th percentile of this distribution. The company therefore chooses α and θ so that these values match their observed mean of 2,300 and their observed 95th percentile of 6,700. Which of the following values should they choose for α, and what should be the corresponding value of θ?

(i) 1.21341

(ii) 1.38071

(iii) 1.87386

(iv) 2.43221

The mean of a paralogistic distribution is $\theta\frac{\Gamma\left(1+\frac{1}{\alpha}\right)\Gamma\left(\alpha-\frac{1}{\alpha}\right)}{\Gamma(\alpha)}$. We therefore have the equations

$$\theta\frac{\Gamma\left(1+\frac{1}{\alpha}\right)\Gamma\left(\alpha-\frac{1}{\alpha}\right)}{\Gamma(\alpha)} = 2300$$
$$\left(\frac{1}{1+\left(\frac{6700}{\theta}\right)^{\alpha}}\right)^{\alpha} = 0.05$$

The first equation gives

$$\theta = \frac{2300\Gamma(\alpha)}{\Gamma\left(1+\frac{1}{\alpha}\right)\Gamma\left(\alpha-\frac{1}{\alpha}\right)}$$

so we have the following:

α	θ	$\left(\frac{1}{1+\left(\frac{6700}{\theta}\right)^{\alpha}}\right)^{\alpha}$
(i) 1.21341	983.36	37175.565
(ii) 1.38071	1629.71	19.999
(iii) 1.87386	2767.52	2.550
(iv) 2.43221	3261.71	1.994

So the correct values are $\alpha = 1.38071$ and $\theta = 1629.71$.