# ACSC/STAT 3703, Actuarial Models I (Further Probability with Applications to Actuarial Science)
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Toby Kenney

Homework Sheet 3
Model Solutions

## Basic Questions

1. *Construct a distribution which is a special case of all distributions in the transformed beta family. How many parameters does it have?*

   The transformed $\beta$ has 4 parameters $\alpha$, $\gamma$, $\theta$ and $\tau$. The special cases in the transformed $\beta$ family involve the conditions $\alpha = 1$, $\gamma = 1$, $\tau = 1$, $\alpha = \gamma$ and $\tau = \gamma$. These are all satisfied if $\alpha = \gamma = \tau = 1$. The distribution with these values is therefore a special case of all distributions in the transformed beta family. It's pdf is

   $$f(x) = \left(\frac{\Gamma(1+1)}{\Gamma(1)\Gamma(1)}\right)\frac{1\left(\frac{x}{\theta}\right)^{1\times 1}}{x\left(1+\left(\frac{x}{\theta}\right)^{1}\right)^{1+1}} = \frac{2\theta}{(\theta+x)^2}$$

   It has one parameter.

2. *If $X$ follows a log-logistic distribution with $\gamma = 3$ and $\theta = 400$, what is the distribution of $\frac{1}{X}$?*

   The CDF of $X$ is given by $F_X(x) = \frac{1}{1+\left(\frac{x}{400}\right)^{-3}}$ we therefore have that the survival function of $\frac{1}{X}$ is

   $$S_{\frac{1}{X}}(x) = F_X\left(\frac{1}{x}\right) = \frac{1}{1+(400x)^3}$$

   so the CDF is

   $$F_{\frac{1}{X}}(x) = 1 - \frac{1}{1+(400x)^3} = \frac{(400x)^3}{1+(400x)^3} = \frac{1}{1+(400x)^{-3}}$$

   so $\frac{1}{X}$ has a log-logistic distribution with $\alpha = 3$ and $\theta = \frac{1}{400}$.

3. *What is the limit of a generalised Pareto distribution as $\alpha \to \infty$ and $\theta \to \infty$ with $\frac{\theta}{\alpha} \to \xi$?*

   **Solution 1:**

   The density function of the generalised Pareto distribution is

   $$f(x) = \left(\frac{\Gamma(\alpha+\tau)}{\Gamma(\alpha)\Gamma(\tau)}\right)\frac{\left(\frac{x}{\theta}\right)^{\tau}}{x\left(1+\frac{x}{\theta}\right)^{\alpha+\tau}}$$

If we substitute $\theta = \alpha\xi$, this becomes

$$f(x) = \left(\frac{\Gamma(\alpha+\tau)}{\Gamma(\alpha)\Gamma(\tau)}\right)\frac{1}{x}\left(\frac{\left(\frac{x}{\xi\alpha}\right)}{\left(1+\frac{x}{\xi\alpha}\right)}\right)^{\tau}\frac{1}{\left(1+\frac{x}{\xi\alpha}\right)^{\alpha}}$$

As $\alpha \to \infty$, we have $\frac{\Gamma(\alpha+\tau)}{\Gamma\alpha} \to \alpha^{\tau}$, and $\left(1+\frac{x}{\xi\alpha}\right)^{\alpha} \to e^{\frac{x}{\xi}}$. Substituting these into the formula gives

$$f(x) = \left(\frac{\alpha^{\tau}}{\Gamma(\tau)}\right)\frac{\left(\frac{x}{\xi\alpha}\right)^{\tau}}{x\left(1+\frac{x}{\xi\alpha}\right)^{\tau}}e^{-\frac{x}{\xi}} = \frac{\left(\frac{x}{\xi}\right)^{\tau}}{\Gamma(\tau)x\left(1+\frac{x}{\xi\alpha}\right)^{\tau}}e^{-\frac{x}{\xi}}$$

Finally, as $\alpha \to \infty$, we have $\left(1+\frac{x}{\xi\alpha}\right)^{\tau} \to 1$, so in the limit

$$f(x) = \frac{\left(\frac{x}{\xi}\right)^{\tau}}{\Gamma(\tau)x}e^{-\frac{x}{\xi}}$$

This distribution is a gamma distribution with $\theta = \xi$ and $\alpha = \tau$.

**Solution 2:**

We know that the limit of the transfomed beta distribution as $\alpha \to \infty$ and $\theta \to \infty$ with $\frac{\theta}{\alpha} \to \xi$ is a transformed gamma distribution with the same $\gamma$ parameter, with $\alpha = \tau$ and with $\theta = \xi$. The limit of the generalised Pareto distribution, which is the special case $\gamma = 1$ is therefore the special case of the transformed gamma distribution, which is the gamma distribution.

4. *Calculate the skewness of a linear exponential distribution with pdf*

$$f_X(x) = \frac{p(x)e^{r(\theta)x}}{q(\theta)}$$

We have that $\mu_2'(\theta) = \mu(\theta)^2 + \frac{\mu'(\theta)}{r'(\theta)}$. To calculate the raw third moment, we note that if we differentiate

$$q(\theta)\mu_2'(\theta) = \int_a^b x^2 p(x)e^{r(\theta)x}\,dx$$

with respect to $\theta$, we get

$$q'(\theta)\mu_2'(\theta) + q(\theta)\frac{d\mu_2'(\theta)}{d\theta} = r'(\theta)\int_a^b x^3 p(x)e^{r(\theta)x}\,dx = r'(\theta)q(\theta)\mu_3'(\theta)$$

Substituting in our expression for $\mu_2'(\theta)$, we get that

$$\frac{d\mu_2'(\theta)}{d\theta} = 2\mu'(\theta)\mu(\theta) + \frac{\mu''(\theta)}{r'(\theta)} - \frac{\mu'(\theta)r''(\theta)}{r'(\theta)^2}$$

Now we have that $\mu_3(\theta) = \mu_3'(\theta) - 3\mu(\theta)\mu_2'(\theta) + 2\mu(\theta)^3$. Substituting in gives

$$\mu_3(\theta) = \frac{q'(\theta)\mu_2'(\theta)}{r'(\theta)q(\theta)} + \left(\frac{2\mu'(\theta)\mu(\theta)}{r'(\theta)} + \frac{\mu''(\theta)}{r'(\theta)^2} - \frac{\mu'(\theta)r''(\theta)}{r'(\theta)^3}\right) - 3\mu(\theta)\mu_2'(\theta) + 2\mu(\theta)^3$$

Recalling that $\mu(\theta) = \frac{q'(\theta)}{r'(\theta)q(\theta)}$, we get

$$\begin{aligned}\mu_3(\theta) &= 3\mu(\theta)\mu_2'(\theta) - 2\mu(\theta)^3 + \frac{\mu''(\theta)}{r'(\theta)^2} - \frac{\mu'(\theta)r''(\theta)}{r'(\theta)^3} - 3\mu(\theta)\mu_2'(\theta) + 2\mu(\theta)^3 \\ &= \frac{\mu''(\theta)}{r'(\theta)^2} - \frac{\mu'(\theta)r''(\theta)}{r'(\theta)^3}\end{aligned}$$

The skewness is $\frac{\mu_3(\theta)}{\mu_2(\theta)^{\frac{3}{2}}}$ which when we substitute the above expression becomes:

$$\frac{\frac{\mu''(\theta)}{r'(\theta)^2} - \frac{\mu'(\theta)r''(\theta)}{r'(\theta)^3}}{\left(\frac{\mu'(\theta)}{r'(\theta)}\right)^{\frac{3}{2}}} = \frac{\mu''(\theta)}{\mu'(\theta)^{\frac{3}{2}}\sqrt{r'(\theta)}} - \frac{r''(\theta)}{\sqrt{\mu'(\theta)}r'(\theta)^{\frac{3}{2}}} = \sqrt{\frac{q(\theta)}{q'(\theta)}}\left(\frac{\mu''(\theta)}{\mu'(\theta)} - \frac{r''(\theta)}{r'(\theta)}\right)$$

## Standard Questions

5. *What is the limiting distribution of an inverse transformed gamma distribution as $\alpha \to \infty$, $\tau \to 0$ and $\theta \to 0$, with $\frac{\sqrt{\theta^\tau}}{\tau} \to \sigma$ and $\frac{\theta^\tau \alpha - 1}{\tau} \to \mu$?*

   Recall that an inverse transformed gamma with parameters $\alpha$, $\tau$ and $\theta$ is the inverse of a transformed gamma distribution with parameters $\alpha$, $\tau$ and $\frac{1}{\theta}$. Therefore, as $\alpha \to \infty$, $\tau \to 0$ and $\theta \to 0$, with $\frac{\sqrt{\theta^\tau}}{\tau} \to \sigma$ and $\frac{\theta^\tau \alpha - 1}{\tau} \to \mu$, the limiting distribution is the inverse of the limiting distribution of a transformed gamma. That is, a log-normal distribution with parameters $\mu$ and $\sigma$. The inverse of this is a log-normal distribution with parameters $-\mu$ and $\sigma$.

6. *An insurance company is modelling claim size by a distribution from a linear exponential family with one parameter $\theta$, with density function*

$$f_X(x) = \frac{p(x)e^{r(\theta)x}}{q(\theta)}$$

   *on its support, $[0, \infty)$. The company wants the mean of the distribution to be $\theta$, and the variance to be $\frac{\theta^2}{3}$ for all values of $\theta$. What is the pdf of the distribution? [Hint: $p(x)$ will be a quadratic function of $x$. Try evaluating the integral for $p(x) = x^2$, $p(x) = x$ and $p(x) = 1$ to find the correct function for $p(x)$.]*

We know that the mean of a distribution in the linear exponential family is $\frac{q'(\theta)}{r'(\theta)q(\theta)}$, and the variance is $\frac{\mu'(\theta)}{r'(\theta)}$. Since we want $\mu(\theta) = \theta$, and the variance to equal $\frac{\theta^2}{3}$, we get $\frac{1}{r'(\theta)} = \frac{\theta^2}{3}$, we need $r'(\theta) = \frac{3}{\theta^2}$, and integrating, we get $r(\theta) = C - \frac{3}{\theta}$. The equation for the mean then gives $\frac{q'(\theta)}{q(\theta)} = r'(\theta)\theta = \frac{3}{\theta}$. Integrating this gives $\log(q(\theta)) = K + 3\log(\theta)$, or $q(\theta) = C_2\theta^3$.

Finally, for this to be a distribution, we need $\int_0^\infty p(x)e^{r(\theta)x}dx = q(\theta)$. Multiplying be a constant if necessary, we can ensure that $C_2 = 1$, so $\int_0^\infty p(x)e^{(C-\frac{3}{\theta})x}dx = \theta^3$. For $p = x^2$, this integral is $\int_0^\infty x^2e^{(C-\frac{3}{\theta})x}dx = \frac{1}{2}\left(\frac{\theta}{3-C\theta}\right)^3$. For the integral to converge, we must have $C\theta < 3$ for all values of $\theta$ in the interval $[0,\infty)$, which means that $C \leqslant 0$. We see that $C = 0$ gives a solution with $p(x) = 54x^2$. The distribution is therefore

$$f(x) = \frac{54x^2e^{-3\frac{x}{\theta}}}{\theta^3}$$