ACSC/STAT 3703, Actuarial Models I (Further Probability with Applications to Actuarial Science) Winter 2015

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Homework Sheet 5
Model Solutions

Basic Questions

1. *An insurance company has an insurance policy where the loss amount follows a gamma distribution with $\alpha = 2$ and $\theta = 1000$. Calculate the expected payment per claim if the company introduces a deductible of d.*

   If the company introduces a deductible of d, the expected payment per loss is given by

$$\frac{\int_d^\infty (x-d)xe^{-\frac{x}{1000}}\,dx}{1000^2} = \frac{\int_d^\infty x^2e^{-\frac{x}{1000}}\,dx - d\int_d^\infty xe^{-\frac{x}{1000}}}{1000^2}dx$$

We integrate by parts

$$\int_d^\infty x^2e^{-\frac{x}{1000}}\,dx = \left[-1000x^2e^{-\frac{x}{1000}}\right]_d^\infty + 2000\int_d^\infty xe^{-\frac{x}{1000}}\,dx = 1000d^2e^{-\frac{d}{1000}} + 2000\int_d^\infty xe^{-\frac{x}{1000}}\,dx$$

and

$$\int_d^\infty xe^{-\frac{x}{1000}}\,dx = \left[-1000xe^{-\frac{x}{1000}}\right]_d^\infty + 1000\int_d^\infty e^{-\frac{x}{1000}}\,dx = 1000de^{-\frac{d}{1000}} + 1000^2e^{-\frac{d}{1000}}$$

We therefore have that the expected payment per loss is

$$\frac{1000d^2e^{-\frac{d}{1000}} + (2000-d)(1000de^{-\frac{d}{1000}} + 1000^2e^{-\frac{d}{1000}})}{1000^2} =$$
$$\frac{(d^2 + (2000-d)(d+1000))e^{-\frac{d}{1000}}}{1000} = (d+2000)e^{-\frac{d}{1000}}$$

The probability that a loss results in a claim is

$$\frac{\int_d^\infty xe^{-\frac{x}{1000}}\,dx}{1000^2} = \frac{[-1000xe^{-\frac{x}{1000}}]_d^\infty + 1000\int_d^\infty e^{-\frac{x}{1000}}\,dx}{1000^2} = \frac{(d+1000)e^{-\frac{d}{1000}}}{1000}$$

The expected payment per claim is therefore given by

$$\frac{(d+2000)e^{-\frac{d}{1000}}}{\left(\frac{(d+1000)e^{-\frac{d}{1000}}}{1000}\right)} = 1000\left(\frac{d+2000}{d+1000}\right)$$

2. *The severity of a loss on a car insurance policy follows a Pareto distribution with $\alpha = 3$ and $\theta = 3000$. Calculate the loss eliminatrion ratio of a deductible of \$1,000.*

   The expected value of a random loss is $\frac{\theta}{\alpha-1} = 1500$. The expected payment per loss with the deductible of \$1,000 is $\int_{1000}^{\infty}\left(1+\frac{x}{3000}\right)^{-3}dx$. Making the substitution, $u = 1+\frac{x}{3000}$, we get that the expected payment per loss is

$$3000\int_{1+\frac{1000}{3000}}^{\infty} u^{-3}du = 3000\left[-\frac{1}{2u^2}\right]_{\frac{4}{3}}^{\infty} = \frac{3000}{2\left(\frac{4}{3}\right)^2} = \frac{27000}{32}$$

   The loss elimination ratio is therefore

$$1-\frac{27000}{32\times 1500} = \frac{7}{16} = 43.75\%$$

3. *An insurance company has a policy where losses follow a Weibull distribution with $\tau = 0.5$ and $\theta = 6000$. The company's risk management division decides that the TVaR at the 95% level, for this policy needs to be reduced to \$75,000. What policy limit should the company put on the policy to achieve this?*

   (i) \$84,400 (ii)\$96,300 (iii) \$122,000 (iv) \$147,000

   The survival function of this Weibull distribution is

$$S(x) = e^{-\left(\frac{x}{\theta}\right)^{\tau}} = e^{-\left(\frac{x}{6000}\right)^{0.5}}$$

   Setting this equal to 0.05 gives

$$\begin{aligned} e^{-\left(\frac{x}{6000}\right)^{0.5}} &= 0.05 \\ \left(\frac{x}{6000}\right)^{0.5} &= \log(20) \\ x &= 6000\log(20)^2 \\ &= 53846.47 \end{aligned}$$

   The TVaR is then the conditional expected value of X given that X is above the 95th percentile. For a policy limit of u, this is

$$53846.47 + \frac{\int_{53846.47}^{u} e^{-\left(\frac{x}{6000}\right)^{0.5}}dx}{0.05}$$

Letting $\nu = \left(\frac{x}{6000}\right)^{0.5}$, we have $\frac{d\nu}{dx} = \frac{0.5x^{-0.5}}{6000^{0.5}} = \frac{\nu^{-1}}{12000}$. We therefore have that the TVaR is

$$53846.47 + 20\int_{\log(20)}^{\left(\frac{u}{6000}\right)^{0.5}} 12000\nu e^{-\nu} d\nu$$

$$= 53846.47 + 240000[-\nu e^{-\nu}]_{\log(20)}^{\sqrt{\frac{u}{6000}}} + 240000\int_{\log(20)}^{\sqrt{\frac{u}{6000}}} e^{-\nu} d\nu$$

$$= 53846.47 + 240000[-\nu e^{-\nu}]_{\log(20)}^{\sqrt{\frac{u}{6000}}} + 240000[-e^{-\nu}]_{\log(20)}^{\sqrt{\frac{u}{6000}}}$$

$$= 53846.47 + 12000(\log(20)+1) - 240000\sqrt{\frac{u}{6000}}e^{-\sqrt{\frac{u}{6000}}} - 240000e^{-\sqrt{\frac{u}{6000}}}$$

Setting this equal to \$75,000 gives

$$53846.47 + 12000(\log(20)+1) - 240000\sqrt{\frac{u}{6000}}e^{-\sqrt{\frac{u}{6000}}} - 240000e^{-\sqrt{\frac{u}{6000}}} = 75000$$

$$-240000e^{-\sqrt{\frac{u}{6000}}}(1+\sqrt{\frac{u}{6000}}) = 26795.26$$

$$e^{-\sqrt{\frac{u}{6000}}}(1+\sqrt{\frac{u}{6000}}) = \frac{26795.26}{240000} = 0.11165$$

The options given lead to the following TVaR:

u	$e^{-\sqrt{\frac{u}{6000}}}(1+\sqrt{\frac{u}{6000}})$
(i) 84,400	0.11166
(ii)96,300	0.09112
(iii) 122,000	0.06064
(iv) 147,000	0.04216

So (i) \$84,400 is the appropriate policy limit.

Standard Questions

4. *For a certain insurance policy, losses follow a gamma distribution with* $\alpha = 7$ *and* $\theta = 3,000$. *The deductible is set to achieve a loss elimination ratio of 20%.*

   (a) Calculate the deductible

   (i) 1500

   (ii) 2700

(iii) 3300

(iv) 4200

The expected loss is $7\times 3000 = 21000$. With the deductible d, the expected payment per loss is

$$\frac{\int_d^\infty (x-d)x^6 e^{-\frac{x}{3000}}\,dx}{3000^7\Gamma(7)}$$

Making the substitution $y = \frac{x}{3000}$, this expected payment per loss is

$$\frac{\int_{\frac{d}{3000}}^\infty (3000y-d)y^6 e^{-y}dy}{\Gamma(7)}$$

We need to evaluate this integral. Integrating by parts, we get

$$\int_a^\infty y^n e^{-y}dy = \left[-y^n e^{-y}\right]_a^\infty + n\int_a^\infty y^{n-1}e^{-y}dy = a^n e^{-a} + n\int_a^\infty y^{n-1}e^{-y}dy$$

So the expected payment per loss is

$$\left(\frac{3000a^7}{6!} + (21000-d)\left(\frac{a^6}{6!}+\frac{a^5}{5!}+\frac{a^4}{4!}+\frac{a^3}{3!}+\frac{a^2}{2!}+a+1\right)\right)e^{-a}$$

where $a = \frac{d}{3000}$.

$$3000\left(\frac{a^7}{6!} + (7-a)\left(\frac{a^6}{6!}+\frac{a^5}{5!}+\frac{a^4}{4!}+\frac{a^3}{3!}+\frac{a^2}{2!}+a+1\right)\right)e^{-a} = 16800$$

$$\left(\frac{a^6}{6!}+\frac{2a^5}{5!}+\frac{3a^4}{4!}+\frac{4a^3}{3!}+\frac{5a^2}{2!}+6a+7\right)e^{-a} = \frac{16800}{3000} = 5.6$$

We substitute the options to get:

d	$\left(\frac{a^6}{6!}+\frac{2a^5}{5!}+\frac{3a^4}{4!}+\frac{4a^3}{3!}+\frac{5a^2}{2!}+6a+7\right)e^{-a}$
(i) 1500	6.50
(ii) 2700	6.10
(iii) 3300	5.90
(iv) 4200	5.60

So the deductible should be \$4,200.

(b) Two years later, there has been uniform inflation of 10%, and the company is considering changing the deductible. What is the new loss elimination for the current deductible after this 10% inflation?

After 10% inflation, the loss distribution is a gamma distribution with $\alpha = 7$ and $\theta = 3300$. The expected loss is therefore 23100, and the expected loss with the deductible is $3300\left(\frac{a^6}{6!}+\frac{2a^5}{5!}+\frac{3a^4}{4!}+\frac{4a^3}{3!}+\frac{5a^2}{2!}+6a+7\right)e^{-a}$, where $a = \frac{d}{3300} = 1.272727$. This gives $3300\left(\frac{a^6}{6!}+\frac{2a^5}{5!}+\frac{3a^4}{4!}+\frac{4a^3}{3!}+\frac{5a^2}{2!}+6a+7\right)e^{-a} = 18900.21$ and the loss elimination ratio is $1-\frac{18900.21}{23100} = 18.18\%$.

5. *For a certain insurance policy, losses follow an inverse exponential distribution with $\theta = \frac{1}{2000}$. There is currently a deductible of \$1000, a policy limit of \$500,000, and coinsurance where the insurance pays 80% of the loss above \$1000 and below \$500,000 (so the maximum total payment is \$399,200).*

   (a) Calculate the expected payment per loss. [You may use the approximation $e^{-y} \approx 1 - y$ for small values of y.]

   The survival function of the inverse exponential is $S(x) = 1 - e^{-\frac{1}{2000x}}$ so the expected payment per loss is

$$\int_{1000}^{500000} 0.8(1 - e^{-\frac{1}{2000x}})dx = 399,200 - 0.8\int_{1000}^{500000} e^{-\frac{1}{2000x}}\,dx$$

   We make the substitution $y = \frac{1}{2000x}$ to get

$$\int_{1000}^{500000} e^{-\frac{1}{2000x}}\,dx = \frac{1}{2000}\int_{10^{-9}}^{5\times10^{-7}} y^{-2}e^{-y}dy \approx \frac{1}{2000}\int_{10^{-9}}^{5\times10^{-7}} y^{-2} - y^{-1}dy = \frac{1}{2000}\left[\log(y) - \frac{1}{y}\right]_{10^{-9}}^{5\times10^{-7}}$$
$$= 500000 - 1000 + \frac{\log(5\times10^{-7}) - \log(10^{-9})}{2000} = 499000 - \frac{\log(500)}{2000}$$

   so the expected payment per loss is

$$399200 - 399200 + \frac{\log(500)}{2000} = \$0.003107304$$

   (b) The insurance company determines that the following actions will all attract more customers:

   (i) removing the deductible.

   (ii) increasing the proportion payed by the insurance to 85%

   (iii) increasing the policy limit to \$1,000,000.

   Which results in the smallest increase to the expected payment per loss?

   Under (i), the new expected payment per loss is

$$400000 - \frac{0.8}{2000}\int_{10^{-9}}^{\infty} y^{-2}e^{-y}dy$$

   Using the Taylor expansion, we get

$$\int_{10^{-9}}^{\infty} y^{-2}e^{-y}dy = \int_{10^{-9}}^{\infty} y^{-2} - y^{-1} + \frac{1}{2} - \frac{y}{6} + \frac{y^2}{24} - \cdots dy$$

   We can divide this integral into 2 parts, from 10^{-9} to A and from A to ∞. The first part can be calculated using the Taylor expansion, while the second part we can approximate by $A^{-2}e^{-A}$. The expected payment then becomes

$$\frac{0.8}{2000}\left(\log\left(\frac{A}{10^{-9}}\right)-\frac{A}{2}+\frac{A^2}{12}-\frac{A^3}{72}+\frac{A^4}{480}-\cdots-A^{-2}e^{-A}\right)=\$0.00881$$

Under (ii), the expected payment per loss is multiplied by $\frac{0.85}{0.8}$, so it becomes \$0.003301511.

Under (iii), the expected payment per loss becomes

$$799600-\frac{0.8}{2000}\int_{5\times 10^{-10}}^{5\times 10^{-7}} y^{-2}e^{-y}\,dy=\frac{\log(1000)}{2000}=\$0.003453878$$

Therefore option (ii) is the least expensive.

6. *A certain insurance policy has losses following a Burr distribution with $\gamma = 0.6$, $\alpha = 2$ and $\theta = 4000$. There is a deductible of \$2,000, and the number of claims under the policy follows a negative binomial distribution with $r = 8$ and $\beta = 1.3$. What would the distribution of the number of claims be if the company reduces the deductible to \$1,000?*

With the deductible at \$2,000, the probability that a loss results in a claim is

$$\frac{1}{(1+\left(\frac{x}{\theta}\right)^{\gamma})^{\alpha}}=\frac{1}{(1+\left(\frac{2000}{4000}\right)^{0.6})^2}=0.363004974$$

With the deductible at \$1,000, the probability that a loss results in a claim is

$$\frac{1}{(1+\left(\frac{1000}{4000}\right)^{0.6})^2}=0.485433327$$

Therefore, the probability that a loss which exceeds \$1,000 also exceeds \$2,000 is $\frac{0.363004974}{0.485433327}=0.74779574$. The new claims distribution should have the property that when compounded with a Bernoulli distribution with probability 0.74779574, the result is a negative binomial distribution with $r = 8$ and $\beta = 1.3$. That is, its probability generating function must satisfy $P(0.25220426+0.74779574z)=(1-1.3(z-1))^{-8}$. If we substitute $u = 0.25220426 + 0.74779574z$, we get $z=\frac{u-0.25220426}{0.74779574}$, so we have

$$P(u)=\left(1-1.3\left(\frac{u-0.25220426}{0.74779574}-1\right)\right)^{-8}=(1-1.738442639(u-1))^{-8}$$

So the new distribution is a negative binomial distribution with $r = 8$ and $\beta = 1.738442639$.