ACSC/STAT 3703, Actuarial Models I (Further Probability with Applications to Actuarial Science)
Winter 2015

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Homework Sheet 6
Model Solutions

Basic Questions

1. *Calculate the VaR and TVaR at the 95% level of the following distribution:*

$$f(x) = \begin{cases} \frac{6(5x^4-x^5)}{3125} & \text{for } 0 < x < 5 \\ 0 & \text{otherwise} \end{cases}$$

The cumulative distribution function is

$$F(x) = \frac{6}{3125}\int_0^x 5y^4 - y^5 dy = \frac{6x^5 - x^6}{3125}$$

We set this equal to 0.95 to find the VaR. This is solved by $x = 4.685751$. We calculate the TVaR from

$$\int_{4.685751}^5 \frac{6}{3125}(5x^5-x^6)dx = \frac{6}{3125}\left[\frac{5x^6}{6} - \frac{x^7}{7}\right]_{4.685751}^5 = 25-\frac{150}{7}-\frac{4.685751^6}{625}+\frac{6\times 4.685751^7}{3125\times 7} = 0.239756$$

So,

$$\text{TVaR}(X) = \frac{0.2397562}{0.05} = 4.795125$$

2. *An insurance company observes the following sample of claims:*

2.8, 2.9, 3.4, 3.9, 4.7, 5.4

It uses these to construct a Kernel density model with a uniform kernel with bandwidth 1.

(a) Under this model, what is the probability that a random claim is greater than 4?

The probability that a random claim is greater than 4 is the average of these probabilities for the 6 kernels arising from the 6 data points. That is

$$\frac{1}{6}\left(0 + 0 + \frac{0.4}{2} + \frac{0.9}{2} + \frac{1.7}{2} + 1\right) = \frac{5}{12} = 0.4166666667$$

(b) Under this model, what is the median claim size?

For claim size x in the interval $[3.8, 3.9]$, the survival function is

$$S(x) = \frac{1}{6}\left(0 + \frac{3.9 - x}{2} + \frac{4.4 - x}{2} + \frac{4.9 - x}{2} + \frac{5.7 - x}{2} + 1\right)$$

Setting this equal to $\frac{1}{2}$ gives the equation $3.9 + 4.4 + 4.9 + 5.7 - 4x = 4$, so $x = \frac{14.9}{4} = 3.725$. Since this is outside the interval, the median is not in this interval. Instead, we try the interval $[3.7, 3.8]$. On this interval, we have

$$S(x) = \frac{1}{6}\left(\frac{3.8 - x}{2} + \frac{3.9 - x}{2} + \frac{4.4 - x}{2} + \frac{4.9 - x}{2} + \frac{5.7 - x}{2} + 1\right)$$

Setting this equal to $\frac{1}{2}$ gives the equation $3.8+3.9+4.4+4.9+5.7-5x = 4$, so $x = \frac{18.7}{5} = 3.74$, so the median is 3.74.

Standard Questions

3. *An insurance company models the aggregate losses on a portfolio as following a Pareto distribution with $\theta = 1000000$ and $\alpha = \frac{10000}{N}$ where N is the number of policies issued. The risk management division asks them to ensure that the TVaR at the 99% level on the portfolio is at most 100,000. How many policies can they issue while ensuring this condition holds.*

   (i) 104

   (ii) 203

   (iii) 297

   (iv) 480

   Since θ is a scale parameter, the TVaR is proportional to θ, so we can simplify calculations by setting $\theta = 1$, then rescaling. For $\theta = 1$, the VaR at the 99% level is found by solving

$$\frac{1}{(1+v)^\alpha} = 0.01$$

   which gives $v = 100^{\frac{1}{\alpha}} - 1$. The TVaR is given by $v + \frac{\int_v^\infty \frac{1}{(1+x)^\alpha}\,dx}{0.01} = v + 100\left[\frac{1}{(\alpha-1)(1+x)^{\alpha-1}}\right]_v^\infty = v + \frac{100}{(\alpha-1)(1+v)^{\alpha-1}}$ where v is the VaR. Since $\frac{1}{(1+v)^\alpha} = 0.01$, we get that the TVaR is $v + \frac{1+v}{\alpha-1}$. We want to solve $v + \frac{1+v}{\alpha-1} = 0.1$. This gives $\alpha v + 1 = 0.1(\alpha - 1)$. Substituting $v = 100^{\frac{1}{\alpha}} - 1$, we get

$$\alpha 100^{\frac{1}{\alpha}} = 1.1\alpha - 1.1$$

For $\alpha = \frac{10000}{N}$, we get $\frac{10000}{N}100^{\frac{N}{10000}} - \frac{11000}{N} + 1.1 = 0$, or $10000 \times 100^{\frac{N}{10000}} - 11000 + 1.1N = 0$

We try the values given:

N	$10000 \times 100^{\frac{N}{10000}} - 11000 + 1.1N$
104	-395.0079
203	203.2408
297	792.3848
480	2001.835

The largest answer given is satisfying this condition is (i) $N = 104$.

4. *Losses follow a gamma distribution with $\alpha = 3$. The insurance company uses the standard deviation principle $r(X) = \mu + a\sigma$ and VaR to measure the risk. For this distribution, what value of a gives the same value of the risk as VaR at the 95% level?*

   Since θ is a scale parameter, all values of θ should give the same answer. To simplify the calculations, we set $\theta = 1$. This gives $\mu = 3$ and $\sigma^2 = 3$. We therefore have $r(X) = 3 + \sqrt{3}a$. We want to select a, so that this is equal to the 95th percentile of the distribution. Integrating by parts, we get

$$S(x) = e^{-x}\left(1 + x + \frac{x^2}{2}\right)$$

   We need to solve when this is 0.05. This is solved by $x = 6.295794$. To get the same value of risk, we therefore solve $3 + \sqrt{3}a = 6.295794$, which gives $a = \frac{3.295794}{\sqrt{3}} = 1.902828$.

5. *An insurance company uses a kernel density model with a Gaussian kernel with standard deviation 1. It observes the following sample of 5 claims:*

   1.4, 2.6, 2.7, 3.5, 3.8

   It wants to check that the VaR at the 95% level is at most 5.2. It plans to collect one additional data point. Based on the Kernel density model, what is the largest value of the last data point which will give a VaR of at most 5.2?

   For the VaR to be at most 5.2, the distribution function must satisfy $F(5.2) > 0.95$. Let x be the last sample point collected. The distribution function at 5.2 is given by

$$\frac{1}{6}(\Phi(5.2 - x) + \Phi(5.2 - 1.4) + \Phi(5.2 - 2.6) + \Phi(5.2 - 2.7) + \Phi(5.2 - 3.5) + \Phi(5.2 - 3.8))$$
$$= \frac{1}{6}(\Phi(5.2 - x) + 0.9999 + 0.9953 + 0.9938 + 0.9554 + 0.9192) = \frac{\Phi(5.2 - x)}{6} + \frac{4.8636}{6}$$

We set this equal to 0.95, which gives

$$\Phi(5.2 - x) = 5.7 - 4.8636 = 0.8364$$

This gives $5.2 - x = 0.98$, so the estimated VaR will be less than 5.2 whenever we have $x < 4.22$.