ACSC/STAT 3703, Actuarial Models I (Further Probability with Applications to Actuarial Science)

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Toby Kenney

Homework Sheet 7

Model Solutions

Basic Questions

1. *Individual policy holders are each assigned a risk factor Θ, which follows a Pareto distribution with $\alpha = 2$ and $\theta = 100$. For a policy holder with risk factor $\Theta = \theta$, the size of a claim follows a log-logistic distribution with $\gamma = 2$ and this value of θ. Calculate the probability that a random claim exceeds \$30,000. [Assume policyholders have the same probability of making a claim regardless of their risk factor.]*

   The probability that a random claim exceeds \$30,000 is the expected value of the conditional probability over the distribution of Θ. For a fixed value of $\Theta = \theta$, this probability is $\frac{\theta^2}{\theta^2+30000}$. The expected value of this is therefore

   $$\int_0^\infty \frac{10000}{2(100+\theta)^3}\frac{\theta^2}{\theta^2+30000}d\theta$$

   We calculate

   $$\frac{1}{\theta^2+30000}-\frac{1}{(\theta+100)^2} = \frac{200\theta-20000}{(\theta+100)^2(\theta^2+30000)} = \frac{200(\theta^2-10000)}{(\theta+100)^3(\theta^2+30000)}$$

   So

   $$\begin{aligned}\frac{3}{\theta^2+30000} - \frac{3}{(\theta+100)^2} + \frac{200}{(\theta+100)^3} &= \frac{600(\theta^2-10000)+200(\theta^2+30000)}{(\theta+100)^3(\theta^2+30000)}\\ &= \frac{400\theta^2}{(\theta+100)^3(\theta^2+30000)}\end{aligned}$$

   Therefore, the required probability is

   $$\begin{aligned}\int_0^\infty \frac{75}{\theta^2+30000} - \frac{75}{(\theta+100)^2} + \frac{5000}{(\theta+100)^3}d\theta &= \frac{75}{\sqrt{30000}}\int_0^\infty \frac{1}{1+u^2}du - 0.75 + 0.25\\ &= \frac{75\pi}{2\sqrt{30000}} - 0.5 = 0.1801748\end{aligned}$$

2. *An insurance company divides claims into three intervals: claims less than \$2,000; claims between \$2,000 and \$20,000; and claims larger than*

\$20,000. It uses the following distributions to model claim size on these three intervals:

Interval	*Probability claim is in this interval*	*Distribution of claims in this interval*
$[0, 2000]$	*0.6*	*Uniform*
$[2000, 20000]$	*0.3*	*Gamma,* $\alpha = 3$*,* $\theta = 1200$
$[20000, \infty]$	*0.1*	*Pareto,* $\alpha = 4$*,* $\theta = 1500$

(These distributions are all truncated to their intervals). Calculate the expected value and variance of a random claim.

For the uniform distribution, the expected value is 1000, and the variance is 333333.33333. For the gamma distribution, the expected value is given by

$$\frac{\int_{2000}^{20000} x^3 e^{-\frac{x}{1200}}\,dx}{\int_{2000}^{20000} x^2 e^{-\frac{x}{1200}}\,dx}$$

Integrating by parts, we have

$$\int_a^b x^3 e^{-x} dx = [-x^3 e^{-x}]_a^b + \int_a^b 3x^2 e^{-x} dx$$

Similarly,

$$\int_a^b x^2 e^{-x} dx = [e^{-x}(x^2 + 2x + 2)]_a^b$$

We therefore have that the mean of the truncated gamma distribution is

$$1200\left(3 + \frac{a^3 e^{-a} - b^3 e^{-b}}{(a^2 + 2a + 2)e^{-a} - (b^2 + 2b + 2)e^{-b}}\right) = 4314.076$$

where $a = \frac{2000}{1200}$ and $b = \frac{20000}{1200}$. [The change of variable from x to $\frac{x}{1200}$ introduces a factor of 1200^3 on the top of the fraction and 1200^2 on the bottom.] The second raw moment of the truncated gamma distribution is

$$\frac{\int_{2000}^{20000} x^4 e^{-\frac{x}{1200}}\,dx}{\int_{2000}^{20000} x^2 e^{-\frac{x}{1200}}\,dx}$$

which is

$$1200^2\left(12 + \frac{(a^4 + 4a^3)e^{-a} - (b^4 + 4b^3)e^{-b}}{(a^2 + 2a + 2)e^{-a} - (b^2 + 2b + 2)e^{-b}}\right) = 22131784$$

so the variance is $22131784 - 4314.076^2 = 3520532$.

The Pareto distribution has mean

$$20000 + \int_{20000}^{\infty} \left(\frac{21500}{1500 + x}\right)^4 dx = 20000 + 21500^4 \int_{21500}^{\infty} u^{-4} du = 20000 + \frac{21500}{3}$$

and raw second moment

$$\begin{aligned}
&\frac{21500^4}{1500^4}\int_{20000}^{\infty}\frac{4\times 1500^4x^2}{(1500+x)^5}dx\\
&=4\int_{20000}^{\infty}21500^4(1500+x)^{-3}(1-2\times 1500(1500+x)^{-1}+1500^2(1500+x)^{-2})dx\\
&=4\times 21500^4\left[\frac{(1500+x)^{-2}}{2}-\frac{2\times 1500(1500+x)^{-3}}{3}+\frac{1500^2(1500+x)^{-4}dx}{4}\right]_{20000}^{\infty}\\
&=2\times 21500^2-\frac{8\times 21500\times 1500}{3}+1500^2=840750000
\end{aligned}$$

so the variance is

$$840750000-\frac{81500^2}{9}=102722222$$

The expected value of the overall distribution is therefore $0.6\times 1000+0.3\times 4314.076+0.1\times\frac{81500}{3}=4610.889$. The expected value of the conditional variance is $0.6\times\frac{1000000}{3}+0.3\times 3520532+0.1\times 102722222=11528382$. The expected square of the conditional expectation is $0.6\times 1000^2+0.3\times 4314.076^2+0.1\times\frac{81500^2}{9}=79986153$, so the variance of the conditional expectation is $79986153-4610.889^2=58725856$. The total variance is therefore $58725856+11528382=70254238$.

3. *The number of claims in a given year follows a compound Poisson-Poisson distribution with parameters 4 and 2. Calculate the probability that there are more than 2 claims in a given year.*

The p.g.f. is

$$P(z)=e^{4\left(e^{2(z-1)}-1\right)}$$

Solution 1: Differentiating twice:

$$\begin{aligned}
P'(z)&=8e^{2(z-1)}e^{4\left(e^{2(z-1)}-1\right)}\\
P''(z)&=\left(64e^{4(z-1)}+16e^{2(z-1)}\right)e^{4\left(e^{2(z-1)}-1\right)}
\end{aligned}$$

We have

$$f_X(2)=\frac{P''(0)}{2}=(32e^{-4}+8e^{-2})e^{4\left(e^{-2}-1\right)}=0.05251983$$

Solution 2:

Evaluating the p.g.f. at 0, we get $f_X(0) = e^{4\left(e^{-2}-1\right)} = 0.03147194$.
Using the standard recurrence

$$f_X(k) = e^{-2}\sum_{i=1}^{k} \frac{4i}{k}\frac{2^i}{i!} f_X(k-i)$$

we calculate

$$\begin{aligned} f_X(1) &= 8e^{-2} f_X(0) = 0.03407411 \\ f_X(2) &= e^{-2}\left(4f_X(1) + 8f_X(0)\right) = 0.05251983 \end{aligned}$$

Standard Questions

4. *Individual policy holders are each assigned a risk factor Θ, which follows a Pareto distribution with $\alpha = 2$ and $\theta = 1600$. For a policy holder with risk factor $\Theta = \theta$, the size of a claim follows a Pareto distribution with $\alpha = 2$ and this value of θ. The insurance company buys reinsurance on each policy. This reinsurance pays the portion of any claim above \$50,000. The premium of this reinsurance is set as 1.2 times the expected reinsurance payment. Calculate this premium.*

Conditional on risk factor $\Theta = \theta$, the expected payment on the reinsurance is

$$\int_{50000}^{\infty} \frac{\theta^2}{(\theta+x)^2}\,dx = \left[-\frac{\theta^2}{(x+\theta)}\right]_{50000}^{\infty} = \frac{\theta^2}{\theta+50000}$$

Therefore, the overall expected payment of the reinsurance is

$$\int_0^{\infty} \frac{5120000}{(\theta+1600)^3}\frac{\theta^2}{\theta+50000}\,d\theta$$

We use partial fractions to rewrite

$$\begin{aligned} \frac{\theta^2}{(\theta+1600)^3(\theta+50000)} =& \frac{1600^2}{48400(\theta+1600)^3} - \frac{1600}{48400(\theta+1600)^2} - \frac{80000000}{48400^2(\theta+1600)^2} \\ &+ \frac{50000^2}{48400^3(\theta+1600)} - \frac{50000^2}{48400^3(\theta+50000)} \end{aligned}$$

This gives

$$\begin{aligned} \int_0^{\infty} \frac{5120000}{(\theta+1600)^3}\frac{\theta^2}{\theta+50000}\,d\theta &= 5120000\left(\frac{1}{48400} - \frac{1}{96800} - \frac{50000}{48400^2} + \frac{50000^2}{48400^3}\log\left(\frac{50000}{1600}\right)\right) \\ &= 332.1965 \end{aligned}$$

The premium is 1.2 times this or \$398.62.

5. *The number of claims from 1200 policies in one year follows a compound Poisson-truncated ETNB distribution with the Poisson distribution having $\lambda = 0.4$ and the truncated ETNB distribution having $\beta = 1.5$ and $r = -0.4$.*

(a) The following year The number of policies increases to 2100. Calculate the probability that the number of claims the following year is exactly 2.

The new distribution is a compound Poisson-truncated ETNB with $\lambda = 0.7$, $\beta = 1.5$ and $r = -0.4$.

Since the secondary distribution is truncated, the compound distribution is zero only if the Poisson distribution is zero. The probability of this is $e^{-0.7} = 0.4965853$.

For the secondary distribution we have $p_1^T = \frac{r\beta}{(1+\beta)^{r+1}-(1+\beta)} = \frac{0.6}{2.5-2.5^{0.6}} = 0.782128$. We also have $a = \frac{\beta}{1+\beta} = \frac{3}{5}$ and $b = -1.4a = -\frac{21}{25}$. This gives $p_2 = \left(\frac{3}{5} - \frac{21}{50}\right) p_1 = \frac{9}{50} p_1 = 0.140783$

Using the standard recurrence

$$f_X(k) = \sum_{i=1}^{k} \frac{0.7i}{k} p_i f_X(k-i)$$

we calculate

$$f_X(1) = 0.7 \times 0.782128 e^{-0.7} = 0.2718753$$
$$f_X(2) = 0.35 \times 0.782128 \times 0.2718753 + 0.7 \times 0.140783 e^{-0.7} = 0.123362$$

(b) The company wants to ensure that the number of claims is at most 2 with probability at least 0.9. How many policies can it issue while maintaining this condition?

(i) 969

(ii) 1356

(iii) 1760

(iv) 1987

If the primary distribution is Poisson with parameter λ, the recurrence gives

$$f_X(1) = 0.782128 \lambda e^{-\lambda}$$
$$f_X(2) = 0.782128^2 \frac{\lambda^2}{2} e^{-\lambda} + 0.140783 \lambda e^{-\lambda}$$

Therefore, the probability that the number of claims is at most 2 is

$$e^{-\lambda}\left(1+0.782128\lambda+0.140783\lambda+0.782128^{2}\frac{\lambda^{2}}{2}\right)=e^{-\lambda}\left(1+0.922911\lambda+0.3058621\lambda^{2}\right)$$

We find the number of policies by setting this equal to 0.9, which gives the solution $\lambda = 0.6625$. Since $\lambda = \frac{N}{3000}$, where N is the number of policies, we get $N = 3000 \times 0.6625 = 1987.5$, it can issue at most 1987 policies while maintaining this condition.