ACSC/STAT 3703, Actuarial Models I

WINTER 2023

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Homework Sheet 4

Due: Monday 13th February: 11:30

Note: This homework assignment is only valid for WINTER 2023. If you find this homework in a different term, please contact me to find the correct homework sheet.

Basic Questions

1. A distribution has survival function

$$S(x) = \begin{cases} 1 & \text{if } x \leqslant 1 \\ x^{-1}2^{-n} & \texttt{if } 2^n \leqslant x \leqslant 2^{n+1} - \frac{1}{n} \\ \frac{4^n(n-1)(1+(n-1)(2^n-x))+2^n((n-1)(2^n-x)-1)}{16^n(n-1)-8^n} & \texttt{if } 2^n - \frac{1}{n-1} \leqslant x \leqslant 2^n \end{cases}$$

where $n \geqslant 0$. How does the tail weight of this distribution compare to that of a Weibull distribution with $\tau = 0.5$ and $\theta = 1$, when tail-weight is assessed by

(a) Asymptotic behaviour of hazard rate.

(b) Existence of moments.

2. Which coherence properties are satisfied by the following measure of risk?

$$\rho(X) = \sup_x xP(X > x)$$

Give a proof or a counterexample for each property.

3. Calculate the TVaR at the 95% level of a distribution with survival function $S_X(x) = xe^{-x^2}$ for $x > 1$. [You may need to use numerical methods to find the VaR.]

4. Which of the following density functions with parameters α, β and γ are scale distributions? Which have scale parameters?

(i) $f(x) = Cx^{1-\alpha}(x+1)^{1-\beta}\gamma^{\alpha+\beta}$

(ii) $f(x) = Cx^{\alpha-1}\gamma^{-\alpha}\gamma^{\frac{x}{\gamma}}e^{-\frac{x\log(x)}{\gamma}}$

(iii) $f(x) = C\alpha^2\beta^3\gamma^{-1}(x+\alpha)^{-2}(x+\beta)^{-3}e^{-\frac{x}{\gamma}}$

5. An insurance company observes the following sample of claims (in thousands):

   ```
   0.5 1.4 1.6 2.1 2.8 3.9 5.6
   ```

   They use a kernel density model with the following kernel

   $$f(x) = \begin{cases} 1-|x| & \text{if } |x| < 1 \\ 0 & \text{otherwise} \end{cases}$$

   What is the TVaR at the 95% level of the fitted distribution?

Standard Questions

6. A Pareto distribution with α and $\theta = 1$ has mean $\frac{1}{\alpha-1}$ and variance $\frac{\alpha}{(\alpha-1)^2(\alpha-2)}$. You can simulate n random variables following this Pareto distribution with the command

   ```
   sim=runif(n)^(-1/alpha)-1
   ```

   [This is simulating a uniform distribution then transforming the result.]

   Based on the central limit theorem, if we take the average of a sample of n Pareto random variables, this should approximately follow a normal distribution with mean $\frac{1}{\alpha-1}$ and variance $\frac{\alpha}{n(\alpha-1)^2(\alpha-2)}$. Plot the distribution of this sample average for $\alpha = 10$, $\alpha = 2.5$ and $\alpha = 2.2$, for sample sizes 400, 1000, and 10000, and compare it with the normal distribution.

7. An insurance company uses a kernel density model for losses, using a gamma kernel with a fixed value of $\alpha = 10$ and $\theta = \frac{x}{10}$ for each observed sample x. The largest eight losses in the sample were

   $ 5,031,900
   $ 2,528,000
   $ 2,200,600
   $ 1,511,800
   $ 1,273,400
   $ 1,152,400
   $ 947,800
   $ 789,400

   Using this model, the Var at the 95% level is $2,098,300 and the TVaR at the same level is $2,180,610. How many claims were in the sample?

   [You may find it helpful to use the `pgamma` function in `R` to find the distribution function of a gamma distribution.]