ACSC/STAT 3703, Actuarial Models I

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Homework Sheet 5

Model Solutions

Basic Questions

1. *A distribution of a random loss X has density function*

$$f_X(x) = \begin{cases} \frac{Ce^{-\frac{x}{5}}}{(x+1)(x+2)} & \text{if } x > 0 \\ 0 & \text{if} x \leqslant 0 \end{cases}$$

for some constant C. After two years, there has been 15% inflation, so the loss distribution is now the distribution of $1.15X$. What is the density function for this distribution?

The density function is

$$f_{1.15X}(x)\frac{1}{1.15}f_X\left(\frac{x}{1.15}\right) = \begin{cases} \frac{1.15Ce^{-\frac{x}{5.75}}}{(x+1.15)(x+2.3)} & \text{if } x > 0 \\ 0 & \text{if} x \leqslant 0 \end{cases}$$

2. *Calculate the distribution of X^6 when X follows a Pareto distribution with $\alpha = 3$ and $\theta = 13$.*

The survival function of X is $S_X(x) = \left(\frac{\theta}{\theta+x}\right)^\alpha$, so the survival function of X^6 is

$$S_{X^6}(x) = S_X\left(x^{\frac{1}{6}}\right) = \left(\frac{\theta}{\theta + x^{\frac{1}{6}}}\right)^\alpha$$

3. *Let T be the time until a claim is processed. The moment generating function of T is $M_T(t) = \frac{192}{(3-t)(4-t)^3}$. Inflation is at an annual rate of 5%. What is the variance of the random variable 1.05^T?*

We have

$$\mathbb{E}\left(1.05^T\right) = \mathbb{E}\left(e^{\log(1.05)T}\right) = M_T(\log(1.05)) = \frac{192}{(3-\log(1.05))\left(4-\log(1.05)\right)^3} = 1.05465606939$$

and

$$\mathbb{E}\left(\left(1.05^T\right)^2\right) = \mathbb{E}\left(e^{2\log(1.05)T}\right) = M_T(2\log(1.05)) = \frac{192}{(3-2\log(1.05))\left(4-2\log(1.05)\right)^3} = 1.1131126112$$

Therefore the variance of $(1.05)^T$ is

$$1.1131126112 - 1.05465606939^2 = 0.0008131865$$

4. *X is a mixture of 3 distributions:*

   - *With probability* 0.3*, X follows a gamma distribution with $\alpha = 0.3$ and $\theta = 20$.*
   - *With probability* 0.6*, X follows a Pareto distribution with $\alpha = 5$ and $\theta = 60$.*
   - *With probability* 0.1*, X follows a Weibull distribution with $\theta = 20$ and $\tau = 4$.*

   The moments of these distributions are given in the following table:

	Distribution 1	*Distribution 2*	*Distribution 3*
Probability	*0.3*	*0.6*	*0.1*
μ	*6*	*15*	*18.12805*
μ_2	*120*	75	25.86457
μ_3	4800	2250	−11.47475
μ_4	331200	253125	1838.22388
μ_2'	156	300	*354.49077*
μ_3'	7176	9000	*7352.50021*
μ_4'	473616	540000	*160000.00000*

   (a) What is the coefficient of variation of X?

   We have $\mathbb{E}(X) = 0.3 \times 6 + 0.6 \times 15 + 0.1 \times 18.12805 = 12.612805$ and $\mathbb{E}(X^2) = 0.3\times156+0.6\times300+0.1\times354.490776 = 262.2490776$. Therefore, $\mathrm{Var}(X) = 262.2490776 - 12.612805^2 = 103.166227632$. The coefficient of variation is therefore $\frac{\sqrt{103.166227632}}{12.612805} = 0.805298877704$.

   (b) [bonus] What is the kurtosis of X?

   We have $\mathbb{E}(X^3) = 0.3\times7176+0.6\times9000+0.1\times7352.50021 = 8288.050021$ and $\mathbb{E}(X^4) = 0.3\times473616+0.6\times540000+0.1\times160000.00000 = 482084.8$.

   It follows that

$$\mu_4 = \mu_4' - 4\mu\mu_3' + 6\mu^2\mu_2' - 3\mu^4 = 482084.8 - 4\times12.612805\times8288.050021 + 6\times12.612805^2\times262.2490776 - 3\times12.612805^4 = 238336.489555$$

so the kurtosis is $\frac{238336.489555}{103.166227632^2} = 22.3931627992$.

5. *For a particular claim, the insurance company has observed the following claim sizes:*

   1.1 1.9 3.0 7.3 10.9 12.8 14.8 15.0 25.6 39.2

   Using a kernel smoothing model with a Gaussian kernel with variance 4, calculate the probability that the next claim size is between 14 and 24.

   We compute the following table

x_i	$\frac{14-x_i}{2}$	$\Phi\left(\frac{14-x_i}{2}\right)$	$\frac{24-x_i}{2}$	$\Phi\left(\frac{24-x_i}{2}\right)$	$\Phi\left(\frac{14-x_i}{2}\right) - \Phi\left(\frac{24-x_i}{2}\right)$
1.1	6.45	1.00000	11.45	1.00000	0.00000
1.9	6.05	1.00000	11.05	1.00000	0.00000
3.0	5.50	1.00000	10.50	1.00000	0.00000
7.3	3.35	0.99960	8.35	1.00000	0.00040
10.9	1.55	0.93943	6.55	1.00000	0.06057
12.8	0.60	0.72575	5.60	1.00000	0.27425
14.8	−0.40	0.34458	4.60	1.00000	0.65542
15.0	−0.50	0.30854	4.50	1.00000	0.69146
25.6	−5.80	0.00000	−0.80	0.21186	0.21186
39.2	−12.60	0.00000	−7.60	0.00000	0.00000
					1.89396

   Thus, the probability is $\frac{1.89396}{10} = 0.189396$.

Standard Questions

6. *An insurance company models the claims of an individual (in dollars) as following a Pareto distribution with $\theta = 1000$ and α varying between individuals. For a random individual, α is assumed to follow a Gamma distribution with shape α and scale θ.*

   From the insurer's data, 5% of claims exceed \$700 and 1% of claims exceed \$5,500. Which of the following values of α would achieve this, and what is the corresponding value of θ? Justify your answer.

   (i) $\alpha = 3.48762$.

   (ii) $\alpha = 7.42930$.

   (iii) $\alpha = 11.09824$.

   (iv) $\alpha = 18.14619$.

   For the Pareto distribution with parameters α and $\theta = 1000$, the probability of a claim exceeding \$700 is $\left(\frac{1000}{1700}\right)^\alpha$, and the probability of a claim exceeding \$5,500 is $\left(\frac{1000}{6500}\right)^\alpha$. We therefore have $\mathbb{E}\left(\left(\frac{1000}{1700}\right)^\alpha\right) = 0.05$ and

$\mathbb{E}\left(\left(\frac{1000}{6500}\right)^{\alpha}\right) = 0.01$. We can rewrite this in terms of the moment generating function of the gamma distribution to get $M_\alpha\left(\log\left(\frac{10}{17}\right)\right) = 0.05$ and $M_\alpha\left(\log\left(\frac{2}{13}\right)\right) = 0.01$.

The moment generating function of a gamma distribution with shape α and scale θ is $M(t) = (1-\theta t)^{-\alpha}$. Substituting the equations we have gives

$$\begin{aligned}
\left(1-\log\left(\frac{10}{17}\right)\theta\right)^{-\alpha} &= 0.05 \\
\left(1-\log\left(\frac{2}{13}\right)\theta\right)^{-\alpha} &= 0.01 \\
\log\left(\frac{10}{17}\right)\theta &= 1-20^{\frac{1}{\alpha}} \\
\log\left(\frac{2}{13}\right)\theta &= 1-100^{\frac{1}{\alpha}} \\
100^{\frac{1}{\alpha}}-1 &= \frac{\log(6.5)}{\log(1.7)}\left(20^{\frac{1}{\alpha}}-1\right) \\
\frac{100^{\frac{1}{\alpha}}-1}{\left(20^{\frac{1}{\alpha}}-1\right)} &= 3.52752077025
\end{aligned}$$

We try the given values of α:

(i) $\alpha = 3.48762$. We have $\frac{100^{\frac{1}{\alpha}}-1}{20^{\frac{1}{\alpha}}-1} = \frac{100^{\frac{1}{3.48762}}-1}{20^{\frac{1}{3.48762}}-1} = 2.01736407079$

(ii) $\alpha = 7.42930$. We have $\frac{100^{\frac{1}{\alpha}}-1}{20^{\frac{1}{\alpha}}-1} = \frac{100^{\frac{1}{7.42930}}-1}{20^{\frac{1}{7.42930}}-1} = 1.72892695923$

(iii) $\alpha = 11.09824$. We have $\frac{100^{\frac{1}{\alpha}}-1}{20^{\frac{1}{\alpha}}-1} = \frac{100^{\frac{1}{11.09824}}-1}{20^{\frac{1}{11.09824}}-1} = 1.65968774878$

(iv) $\alpha = 18.14619$. We have $\frac{100^{\frac{1}{\alpha}}-1}{20^{\frac{1}{\alpha}}-1} = \frac{100^{\frac{1}{18.14619}}-1}{20^{\frac{1}{18.14619}}-1} = 1.60943653132$

Thus, we see that (i) $\alpha = 3.48762$ is the closest value of α, and for this value of α, we have

$$\theta = \frac{20^{\frac{1}{3.48762}}-1}{\log(1.7)} = 2.56433478045$$

or using the other equation, we get

$$\theta = \frac{100^{\frac{1}{3.48762}}-1}{\log(6.5)} = 1.46652484521$$

[Actually, the options given do not include the correct value of α, which is 1.367299, and for this value of α, the value of θ is $\theta = \frac{20^{\frac{1}{1.367299}}-1}{\log(1.7)} = 14.9709764395$ or $\theta = \frac{100^{\frac{1}{1.367299}}-1}{\log(1.7)} = 14.9709744233$ with the difference between these values being due to rounding errors.]

7. *The time until failure of a product has hazard rate $\lambda(t) = 2(1-a) + \frac{t^2}{16}$ where a is a measure of the quality of the product, and is modelled as following a distribution with density $f_A(a) = 7.5a^2 - 4.5a + 0.75$ for $0 \leqslant a \leqslant 1$. The product has a two-year waranty. What is the probability that it will be replaced under this waranty?*

   For a given value of a, the probability that the product will last two years is
   $$e^{-\int_0^2 2(1-a)+\frac{t^2}{16}\,dt} = e^{-4(1-a)-\frac{2^3}{3\times 16}\,dt} = e^{4a-\frac{25}{6}}$$

   The probability that a random product lasts for two years is therefore

   $$\begin{aligned}\mathbb{E}(e^{4A-\frac{25}{6}}) = e^{-\frac{25}{6}}M_A(4) = e^{-\frac{25}{6}}\int_0^1 (7.5a^2-4.5a+0.75)e^{4a}\,da\\ = e^{-\frac{25}{6}}\left(\left[(7.5a^2-4.5a+0.75)\frac{e^{4a}}{8}\right]_0^1 - \int_0^1 \frac{(15a-4.5)}{4}e^{4a}\,da\right)\\ = e^{-\frac{25}{6}}\left(\frac{3.75}{4}e^4 - \frac{0.75}{4} - \left[\frac{(15a-4.5)}{16}e^{4a}\,da\right]_0^1 + \int_0^1 \frac{15}{16}e^{4a}\,da\right)\\ = e^{-\frac{25}{6}}\left(\frac{3.75}{4}e^4 - \frac{0.75}{4} - \frac{11.5}{16}e^4 - \frac{4.5}{16} + \frac{15}{64}(e^4-1)\right)\\ = e^{-\frac{25}{6}}\left(0.453125e^4 - 0.703125\right)\\ = 0.372660884528\end{aligned}$$

8. *An insurance company models claims as following a log-normal distribution with $\mu = 4$ and $\sigma^2 = 3$. They want to transform the claims by raising to a power in order to make the kurtosis of the distribution equal to 6. What power should they use? [You may need to use numerical methods to solve the necessary equations.]*

   We have $\mathbb{E}((X^\alpha)^n) = \mathbb{E}(X^{n\alpha}) = M_X(n\alpha)$. For the log-normal distribution, we have $M_X(t) = e^{\mu t+\frac{\sigma^2 t^2}{2}} = e^{4t+1.5t^2}$. We therefore have the following raw moments for X^α

μ	$e^{4\alpha+1.5\alpha^2}$
μ'_2	$e^{8\alpha+6\alpha^2}$
μ'_3	$e^{12\alpha+13.5\alpha^2}$
μ'_4	$e^{16\alpha+24\alpha^2}$

   This gives

   $$\begin{aligned}\mu_4 &= \mu'_4 - 4\mu\mu'_3 + 6\mu^2\mu'_2 - 3\mu^4\\ &= e^{16\alpha+24\alpha^2} - 4e^{16\alpha+15\alpha^2} + 6e^{16\alpha+9\alpha^2} - 3e^{16\alpha+6\alpha^2}\\ &= e^{16\alpha}\left(e^{24\alpha^2} - 4e^{15\alpha^2} + 6e^{9\alpha^2} - 3e^{6\alpha^2}\right)\end{aligned}$$

and

$$\mu_2 = \mu_2' - \mu^2 = e^{8\alpha+6\alpha^2} - e^{8\alpha+3\alpha^2} = e^{8\alpha}\left(e^{6\alpha^2} - e^{3\alpha^2}\right)$$

Therefore, the kurtosis is

$$\frac{e^{16\alpha}\left(e^{24\alpha^2} - 4e^{15\alpha^2} + 6e^{9\alpha^2} - 3e^{6\alpha^2}\right)}{e^{16\alpha}\left(e^{6\alpha^2} - e^{3\alpha^2}\right)^2} = \frac{e^{24\alpha^2} - 4e^{15\alpha^2} + 6e^{9\alpha^2} - 3e^{6\alpha^2}}{e^{12\alpha^2} - 2e^{9\alpha^2} + e^{6\alpha^2}}$$

Letting $x = e^{3\alpha^2}$, this becomes

$$\frac{x^8 - 4x^5 + 6x^3 - 3x^2}{x^4 - 2x^3 + x^2} = \frac{x^6 - 4x^3 + 6x - 3}{x^2 - 2x + 1} = \frac{x^5 + x^4 + x^3 - 3x^2 - 3x + 3}{x - 1} = x^4 + 2x^3 + 3x^2 - 3$$

We set this equal to 6 and solve:

$$x^4 + 2x^3 + 3x^2 - 3 = 6$$
$$x^4 + 2x^3 + 3x^2 - 9 = 0$$

Solving this numerically gives $x = 1.1614443171$, then

$$\alpha = \sqrt{\frac{\log(1.1614443171)}{3}} = 0.223356465149$$