

MATH/STAT 3720, Life Contingencies I  
Winter 2015  
Toby Kenney  
In Class Examples

# Introduction to Life Insurance

## Term Life Insurance Contract

- Policyholder pays regular premiums for the term of the policy, or until death.
- If policyholder dies, insurance pays death benefit.

## Whole Life Insurance Contract

- Policyholder pays regular premiums until death, or an agreed age (usually 80).
- When policyholder dies, insurance pays death benefit.

## Endowment Insurance

- Policyholder pays regular premiums until death, or an agreed future time.
- At death, or the agreed future time, insurance pays agreed benefit.

## Annuities

- Policyholder pays a lump sum
- Insurance pays regular payments until policyholder dies.

## Participating Life Insurance

Amount by which premiums and investment gains exceed liabilities and costs is returned to policyholders in one of a number of ways:

- Cash dividends
- Reduced premiums
- Increased policy value

# Modern Insurance Contracts

- Universal Life Insurance
- Unitized with-profit
- Equity-linked Insurance

# Underwriting

## Underwriting

The process of acquiring information about factors which affect the potential policyholder's risk, and determining an appropriate premium.

Relevant information might include:

- Age
- Sex
- Medical history (personal & family)
- Smoking habits
- Occupation
- Dangerous hobbies

For large sums insured, it may also involve a medical examination.

Policyholders will generally be classified into the following categories:

|                   |   |
|-------------------|---|
| Preferred lives   | very low mortality risk   |
| Normal lives      | may have some higher risk factors                                       |
| Rated lives       | higher than average risk, can be insured for higher premium             |
| Uninsurable lives | very high risk, insurer will not enter insurance contract at any price. |

# Annuities

- Single Premium Deferred Annuity
- Single Premium Immediate Annuity
- Regular Premium Deferred Annuity
- Joint Life Annuity
- Last Survivor Annuity
- Reversionary Annuity

# Other Insurance Contracts

- Income Protection insurance
- Critical Illness insurance
- Long-term care insurance

# Pensions

## Defined Benefit

Annuity upon retirement defined by a formula. For example,

$$\text{Final Salary} \times \text{Years of Service} \times \text{Accrual Rate}$$

Paid for by regular contributions by employer and (usually) employee.

## Defined Contribution

Employer and employee pay pre-determined contribution into fund. Upon retirement, value of fund is available to employee to provide retirement income.



# Actuarial Tasks

- Calculate premiums, taking into account:
  - Probability of death/disability or other claim event
  - Premiums and benefits
  - Investment gains
- Calculate dividends
- Calculate surrender value of policy
- Ensure adequate reserves to cover risk
- Invest money to match future liabilities

## 2.2 The Future Lifetime Random Variable

### Question 1

The probability that a newborn baby will live to age 34 is 0.983. The probability that the same newborn baby will live to age 46 is 0.964. What is the probability that a life aged 34, with the same characteristics, will survive for a further 12 years?

# Properties of Survival Functions

## Properties of Valid Survival Functions

- $S_x(0) = 1$
- $S_x(t)$  is a decreasing function.
- $\lim_{t \rightarrow \infty} S_x(t) = 0$

## Additional Assumptions for Future Lifetime

- $S_x(t)$  is differentiable for all  $t > 0$ .
- $\lim_{t \rightarrow \infty} tS_x(t) = 0$ .
- $\lim_{t \rightarrow \infty} t^2 S_x(t) = 0$ .

# Force of Mortality

## Definition

For a random life, the *force of mortality* at age  $x$  is given by

$$\mu_x = \frac{d}{dt}(S_x(t))|_{t=0}$$

## Other Formulae

- $\mu_x = \frac{f_0(x)}{S_0(x)}$
- $S_0(x) = e^{-\int_0^x \mu_x dx}$
- $S_x(t) = e^{-\int_x^{x+t} \mu_s ds}$

## 2.3 Force of Mortality

### Question 2

Suppose lifetime is modelled as

$$F_0(t) = 1 - \left(1 - \frac{x}{130}\right)^{\frac{1}{4}}$$

Calculate the force of mortality.

## 2.3 Force of Mortality

### Question 3

The Gompertz law of mortality states that  $\mu_x = Bc^x$  for two constants  $B$  and  $c$ , where  $c > 1$ . Calculate the survival function  $S_0(x)$  based on this law.

## 2.4 Actuarial Notation

### Notation

- Survival probability  ${}_t p_x = S_x(t)$
- Mortality probability  ${}_t q_x = F_x(t)$
- Deferred mortality probability  ${}_u|_t q_x = S_x(u) - S_x(t+u)$

### Relations

$${}_t p_x + {}_t q_x = 1$$

$${}_u|_t q_x = {}_u p_x - {}_{u+t} p_x$$

$${}_{u+t} p_x = {}_u p_x {}_t p_{x+u}$$

$$\mu_x = -\frac{1}{{}_x p_0} \frac{d}{{}_x p_0} ({}_x p_0)$$

$$f_x(t) = {}_t p_x \mu_{x+t}$$

$${}_t q_x = \int_0^t {}_s p_x \mu_{x+s} ds$$

## 2.4 Actuarial Notation

### Question 4

Suppose mortality follows a Gompertz law with  $B = 0.005$  and  $c = 1.07$ . Calculate the following:

- (a)  $p_{36}$
- (b)  ${}_3q_{72}$
- (c)  ${}_2|_2q_{29}$



## 2.4 Actuarial Notation

### Question 5

Suppose mortality follows a Gompertz law with  $B = 0.007$  and  $c = 1.06$ .

- (a) Calculate the exact value of  $q_{57}$ , and compare it with  $\mu_{57.5}$ .
- (b) Calculate the exact value of  $q_{87}$  and compare it with  $\mu_{87.5}$ .

### Question 6

Suppose lifetime is modelled as

$$F_0(t) = 1 - \left(1 - \frac{x}{130}\right)^{\frac{1}{4}}$$

- (a) Calculate  $\dot{e}_{39}$ .
- (b) Calculate  $\text{Var}(T_{39})$ .

### Question 7

Suppose lifetime is modelled as

$$F_0(t) = 1 - \left(1 - \frac{x}{130}\right)^{\frac{1}{4}}$$

Calculate  $\ddot{e}_{39:\overline{25}|}$

## 2.6 Curtate Future Lifetime

### Question 8

Future lifetime for a particular life is modelled as following a Gompertz law with  $B = 0.0001$  and  $c = 1.06$ . The life is currently aged 47. What is the probability that the individual's curtate future lifetime is 6?

## 2.6 Curtate Future Lifetime

### Question 9

Future lifetime for a particular life is modelled as following a Gompertz law with  $B = 0.0001$  and  $c = 1.06$ . The life is currently aged 47. What is the individual's curtate expected lifetime?

## 2.6 Curtate Future Lifetime

### Question 10

Suppose lifetime is modelled as

$$F_0(t) = 1 - \left(1 - \frac{x}{130}\right)^{\frac{1}{4}}$$

Calculate  $\ddot{e}_x - (e_x + 0.5)$ .

## 3.2 Life Tables

### Question 11

The following is an extract from a life table:

| $x$ | $l_x$    | $d_x$ |
|-----|----------|-------|
| 40  | 10000.00 | 47.12 |
| 41  | 9952.88  | 49.46 |
| 42  | 9903.42  | 50.93 |
| 43  | 9852.49  | 52.40 |
| 44  | 9800.09  | 55.88 |
| 45  | 9744.21  | 59.94 |
| 46  | 9684.27  | 63.73 |
| 47  | 9621.54  | 67.01 |
| 48  | 9554.53  | 70.66 |
| 49  | 9483.87  | 72.78 |
| 50  | 9411.09  | 74.40 |

Calculate:

- (a)  ${}_6p_{42}$
- (b)  $q_{46}$
- (c)  ${}_4|_2q_{41}$
- (d)  $e_{43:\overline{5}|}$

## 3.2 Life Tables

### Question 12

Compute a lifetable starting with age 30 and radix 10,000, ending at age 40, using a Makeham model of mortality  $\mu_x = A + Bc^x$  with  $A = 0.0002$ ,  $B = 0.0002$  and  $c = 1.06$ . [You may use  $\mu_{x+0.5}$  as an approximation for  $q_x$ .]



## 3.2 Life Tables

### Solution to Question 12

| $x$ | $l_x$    | $d_x$ |
|-----|----------|-------|
| 30  | 10000.00 | 13.83 |
| 31  | 9986.17  | 14.52 |
| 32  | 9971.66  | 15.25 |
| 33  | 9956.41  | 16.02 |
| 34  | 9940.40  | 16.83 |
| 35  | 9923.57  | 17.69 |
| 36  | 9905.88  | 18.60 |
| 37  | 9887.28  | 19.56 |
| 38  | 9867.72  | 20.57 |
| 39  | 9847.14  | 21.64 |
| 40  | 9825.50  | 22.78 |

## 3.3 Fractional Age Assumptions

### Question 13

Under a certain model of mortality, we have  $q_{36} = 0.0004$ . Using the uniform distribution of deaths assumption, what is  ${}_{0.6}q_{36.3}$ ?

## 3.3 Fractional Age Assumptions

### Question 14

The following is an extract from a life table:

| $x$ | $l_x$    | $d_x$ |
|-----|----------|-------|
| 40  | 10000.00 | 47.12 |
| 41  | 9952.88  | 49.46 |
| 42  | 9903.42  | 50.93 |
| 43  | 9852.49  | 52.40 |
| 44  | 9800.09  | 55.88 |
| 45  | 9744.21  | 59.94 |
| 46  | 9684.27  | 63.73 |
| 47  | 9621.54  | 67.01 |
| 48  | 9554.53  | 70.66 |
| 49  | 9483.87  | 72.78 |
| 50  | 9411.09  | 74.40 |

An individual aged 42 and 4 months wishes to purchase a term life insurance contract for 6 years. What is the probability that the individual dies during this contract?

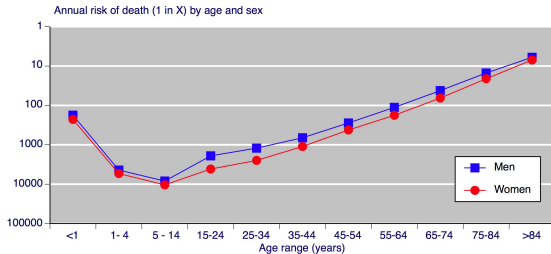
## 3.3 Fractional Age Assumptions

### Question 15

Under a certain model of mortality, we have  $q_{36} = 0.0004$ . Using the constant rate of mortality assumption, what is  ${}_{0.6}q_{36.3}$ ?

## 3.4 National Life Tables

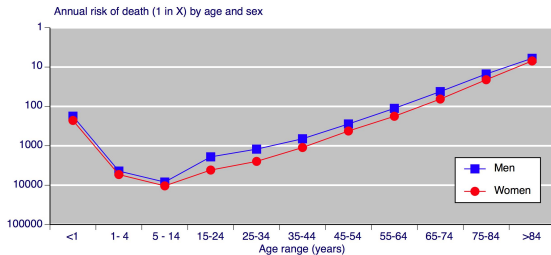
Figure: Mortality Rates by Age from UK Mortality statistics 2005



Source: <http://www.medicine.ox.ac.uk/bandolier/booth/Risk/dyingage.html>

## 3.4 National Life Tables

Figure: Mortality Rates by Age from UK Mortality statistics 2005

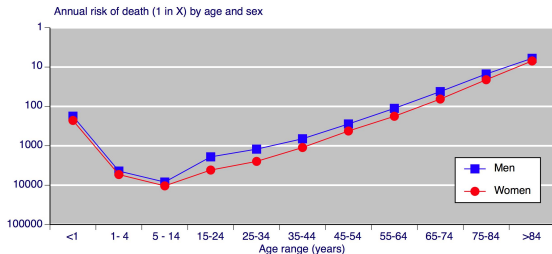


Source: <http://www.medicine.ox.ac.uk/bandolier/booth/Risk/dyingage.html>

- High mortality just after birth.

## 3.4 National Life Tables

Figure: Mortality Rates by Age from UK Mortality statistics 2005

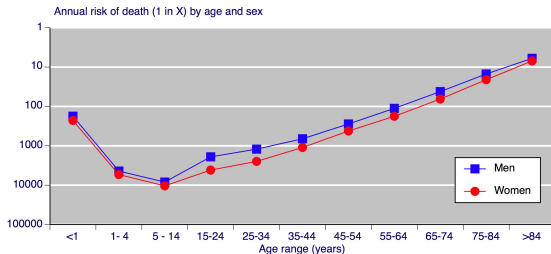


Source: <http://www.medicine.ox.ac.uk/bandolier/booth/Risk/dyingage.html>

- High mortality just after birth.
- Mortality rate drops quickly, continues decreasing till age 10.

## 3.4 National Life Tables

Figure: Mortality Rates by Age from UK Mortality statistics 2005



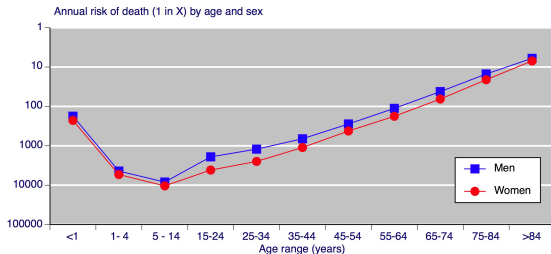
Source: <http://www.medicine.ox.ac.uk/bandolier/booth/Risk/dyingage.html>

- High mortality just after birth.
- Mortality rate drops quickly, continues decreasing till age 10.
- Male and female mortalities diverge significantly in late teens



## 3.4 National Life Tables

Figure: Mortality Rates by Age from UK Mortality statistics 2005

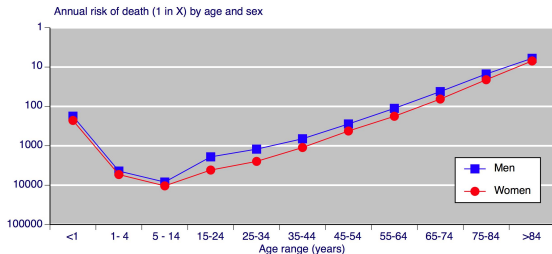


Source: <http://www.medicine.ox.ac.uk/bandolier/booth/Risk/dyingage.html>

- High mortality just after birth.
- Mortality rate drops quickly, continues decreasing till age 10.
- Male and female mortalities diverge significantly in late teens
- Mortality increases after age 10, accident hump in late teens.

## 3.4 National Life Tables

Figure: Mortality Rates by Age from UK Mortality statistics 2005

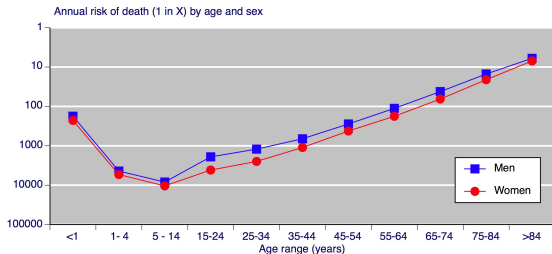


Source: <http://www.medicine.ox.ac.uk/bandolier/booth/Risk/dyingage.html>

- High mortality just after birth.
- Mortality rate drops quickly, continues decreasing till age 10.
- Male and female mortalities diverge significantly in late teens
- Mortality increases after age 10, accident hump in late teens.
- Rates for females are lower than for males.

## 3.4 National Life Tables

Figure: Mortality Rates by Age from UK Mortality statistics 2005



Source: <http://www.medicine.ox.ac.uk/bandolier/booth/Risk/dyingage.html>

- High mortality just after birth.
- Mortality rate drops quickly, continues decreasing till age 10.
- Male and female mortalities diverge significantly in late teens
- Mortality increases after age 10, accident hump in late teens.
- Rates for females are lower than for males.
- Gompertz model fits well for large ages, badly for young ages.

## 3.5 Survival Models for Life Insurance Policyholders

### Question 16

Why are mortality rates lower for policyholders than members of the general population of the same age and sex? Should you purchase a life insurance policy to reduce your mortality?

## 3.5 Survival Models for Life Insurance Policyholders

### Answer to Question 16

- People who buy life insurance are better off financially.
- People who buy life insurance are usually married, often with families.
- In order to buy life insurance individuals must be in good health.

## 3.7 Select and Ultimate Survival Models

### Question 17

Using a model where ultimate mortality follows Makeham's model  $\mu_x = 0.00022 + 0.0000027 \times 1.124^x$ , and the select mortality for an individual selected  $s$  years ago is given by  $\mu_{[x]+s} = 0.9^{2-s} \mu_{x+s}$ . Calculate the probability that an individual currently aged 42, and select at age 41 will live to age 48. [You may use the approximation  $q_x \approx \mu_{x+0.5}$ .]

## 3.9 Select Life Tables

### Question 18

Use the standard select survival model from Question 17 to construct a select life table with  $l_{28} = 100000$ , for ages between 30 and 40 at time of selection.

## 3.9 Select Life Tables

### Answer to Question 18

| $x$ | $l_{[x]}$ | $l_{[x]+1}$ | $l_{[x]+2}$ |
|-----|-----------|-------------|-------------|
| 30  | 99991.33  | 99967.40    | 99939.96    |
| 31  | 99961.50  | 99936.81    | 99908.44    |
| 32  | 99930.70  | 99905.17    | 99875.74    |
| 33  | 99898.82  | 99872.34    | 99841.73    |
| 34  | 99865.72  | 99838.17    | 99806.23    |
| 35  | 99831.26  | 99802.51    | 99769.07    |
| 36  | 99795.26  | 99765.16    | 99730.03    |
| 37  | 99757.53  | 99725.92    | 99688.90    |
| 38  | 99717.86  | 99684.55    | 99645.40    |
| 39  | 99676.01  | 99640.79    | 99599.25    |
| 40  | 99631.71  | 99594.34    | 99550.12    |



## 3.9 Select Life Tables

### Question 19

Using the following select life table

| $x$ | $l_{[x]}$ | $l_{[x]+1}$ | $l_{[x]+2}$ | $l_{[x]+3}$ |
|-----|-----------|-------------|-------------|-------------|
| 25  | 99964.85  | 99938.56    | 99907.27    | 99869.96    |
| 26  | 99921.59  | 99894.99    | 99863.28    | 99825.45    |
| 27  | 99877.76  | 99850.82    | 99818.66    | 99780.23    |
| 28  | 99833.32  | 99805.99    | 99773.33    | 99734.25    |
| 29  | 99788.19  | 99760.44    | 99727.22    | 99687.41    |
| 30  | 99742.30  | 99714.07    | 99680.24    | 99639.61    |
| 31  | 99695.56  | 99666.81    | 99632.28    | 99590.76    |
| 32  | 99647.88  | 99618.54    | 99583.25    | 99540.74    |
| 33  | 99599.16  | 99569.17    | 99533.03    | 99489.41    |

Calculate the probability that an individual aged 27, who has been select for 1 year, will die between at age 31 or 32.

## 3.11 Mortality Trends

### Question 20

The following lifetable applied 8 years ago:

| $x$ | $l_x$    | $d_x$ |
|-----|----------|-------|
| 23  | 10000.00 | 4.59  |
| 24  | 9995.41  | 4.66  |
| 25  | 9990.74  | 4.74  |
| 26  | 9986.01  | 4.82  |
| 27  | 9981.19  | 4.91  |
| 28  | 9976.28  | 5.00  |
| 29  | 9971.28  | 5.11  |
| 30  | 9966.17  | 5.22  |

An insurance company determines that the following reduction factors should be applied to lives of each age:

| age       | 23   | 24   | 25   | 26   | 27    | 28   | 29    | 30   |
|-----------|------|------|------|------|-------|------|-------|------|
| reduction | 0.97 | 0.97 | 0.96 | 0.97 | 0.975 | 0.97 | 0.975 | 0.98 |

Calculate  $e_{25:\overline{5}|}$

## 4.4 Valuation of Life Insurance Benefits

### Notation

|             |   |
|-------------|---|
| $i$         | Effective annual interest rate  |
| $v$         | Annual discount factor $(1 + i)^{-1}$   |
| $\delta$    | Force of interest $\log(1 + i)$   |
| $i^{(p)}$   | Nominal interest rate compounded $p$ times per year   |
| $d$         | Annual discount rate $1 - v$  |
| $d^{(p)}$   | Nominal discount rate compounded $p$ times per year $p(1 - v^{\frac{1}{p}})$  |
| $\bar{A}_x$ | Expected present value of \$1 when a life of present age $x$ dies   |
| $A_x$       | Expected present value of \$1 at the end of the year in which a life of present age $x$ dies                              |
| $A_x^{(m)}$ | Expected present value of \$1 at the end of the period $\frac{1}{m}$ th of a year in which a life of present age $x$ dies |

## 4.4 Valuation of Life Insurance Benefits

### Question 21

Suppose the future lifetime random variable satisfies

$$F_0(x) = 1 - \left(1 - \frac{x}{130}\right)^{\frac{1}{4}}$$

and force of interest is given by  $\delta = 0.04$ .

(a) Calculate  $\bar{A}_{37}$ .

(b) Calculate the variance of the present value of a payment of \$1 immediately at the time the life dies.

### Question 22

Suppose the future lifetime random variable satisfies

$$F_0(t) = 1 - \left(1 - \frac{x}{130}\right)^{\frac{1}{4}}$$

and force of interest is given by  $\delta = 0.05$ .

- (a) Calculate  $A_{43}$ .
- (b) Calculate  $A_{43}^{(12)}$ .

## 4.4 Valuation of Life Insurance Benefits

### Question 23

The following is an excerpt from a life table

| $x$ | $l_x$    | $d_x$ |
|-----|----------|-------|
| 30  | 10000.00 | 3.49  |
| 31  | 9996.51  | 3.60  |
| 32  | 9992.90  | 3.72  |
| 33  | 9989.18  | 3.86  |
| 34  | 9985.32  | 4.00  |
| 35  | 9981.32  | 4.16  |
| 36  | 9977.16  | 4.33  |
| 37  | 9972.82  | 4.52  |
| 38  | 9968.30  | 4.73  |
| 39  | 9963.57  | 4.96  |

You are given that  $A_{40} = 0.1211$ ,  ${}^2A_{40} = 0.03071$ , and  $\delta = 0.06$ . Calculate  $A_{32}$  and the variance of a payment of \$1 when a life currently aged 32 dies.

## 4.4 Valuation of Life Insurance Benefits

### Question 24

The following is an excerpt from a life table.

| $x$ | $l_x$    | $d_x$ |
|-----|----------|-------|
| 32  | 10000.00 | 4.84  |
| 33  | 9995.16  | 4.86  |
| 34  | 9990.30  | 4.89  |
| 35  | 9985.41  | 4.92  |
| 36  | 9980.49  | 4.95  |
| 37  | 9975.54  | 4.98  |
| 38  | 9970.55  | 5.02  |
| 39  | 9965.53  | 5.06  |
| 40  | 9960.48  | 5.10  |

Calculate  $A_{33:\overline{5}|}^1$  at interest rate  $v = 0.96$ .

## 4.4 Valuation of Life Insurance Benefits

### Question 25

The following is an excerpt from a life table.

| $x$ | $l_x$    | $d_x$ |
|-----|----------|-------|
| 30  | 10000.00 | 9.82  |
| 31  | 9990.18  | 10.34 |
| 32  | 9979.84  | 10.95 |
| 33  | 9968.89  | 11.64 |
| 34  | 9957.25  | 12.44 |
| 35  | 9944.80  | 13.37 |
| 36  | 9931.44  | 14.42 |
| 37  | 9917.01  | 15.64 |
| 38  | 9901.37  | 17.05 |
| 39  | 9884.32  | 18.66 |
| 40  | 9865.67  | 20.51 |

Calculate  $A_{30:\overline{10}|}$  at interest rate  $i = 0.07$ .



## 4.4 Valuation of Life Insurance Benefits

### Question 26

The following is an excerpt from a life table.

| $x$ | $l_x$    | $d_x$ |
|-----|----------|-------|
| 30  | 10000.00 | 9.82  |
| 31  | 9990.18  | 10.34 |
| 32  | 9979.84  | 10.95 |
| 33  | 9968.89  | 11.64 |
| 34  | 9957.25  | 12.44 |
| 35  | 9944.80  | 13.37 |
| 36  | 9931.44  | 14.42 |
| 37  | 9917.01  | 15.64 |
| 38  | 9901.37  | 17.05 |
| 39  | 9884.32  | 18.66 |
| 40  | 9865.67  | 20.51 |

Calculate  $4|A_{30:\overline{6}|}$  at interest rate  $i = 0.07$ .

## 4.5 Relating Different Cases of Whole Life Insurance

### Question 27

The following is an excerpt from a life table:

| $x$ | $l_x$    | $d_x$ |
|-----|----------|-------|
| 40  | 10000.00 | 6.63  |
| 41  | 9993.37  | 6.85  |
| 42  | 9986.51  | 7.09  |
| 43  | 9979.42  | 7.36  |
| 44  | 9972.06  | 7.65  |
| 45  | 9964.41  | 7.96  |
| 46  | 9956.45  | 8.31  |
| 47  | 9948.14  | 8.69  |
| 48  | 9939.45  | 9.11  |
| 49  | 9930.34  | 9.56  |
| 50  | 9920.78  | 10.06 |

Force of interest is  $\delta = 0.035$ . Calculate  $\bar{A}_{40:\overline{10}|}^1$  using a uniform distribution of deaths assumption.

## 4.5 Relating Different Cases of Whole Life Insurance

### Question 28

The following is an excerpt from a life table:

| $x$ | $l_x$    | $d_x$  |
|-----|----------|--------|
| 35  | 10000.00 | 54.89  |
| 36  | 9945.11  | 63.26  |
| 37  | 9881.85  | 73.00  |
| 38  | 9808.85  | 84.30  |
| 39  | 9724.54  | 97.38  |
| 40  | 9627.16  | 112.48 |

Calculate  $A^{(12)} \bar{1}_{35:\overline{5}|}$  using:

- (a) Force of interest  $\delta = 0.04$  and the Uniform Distribution of Deaths.
- (b) Force of interest  $\delta = 0.04$  and the claim acceleration method.
- (c) Force of interest  $\delta = 0.12$  and the Uniform Distribution of Deaths.
- (d) Force of interest  $\delta = 0.12$  and the claim acceleration method.

## 4.6 Variable Insurance Benefits

### Question 29

An insurance policy pays a death benefit equal to the number of years the policy has been in force plus one. Using the life table below, calculate the EPV of a policy with a term of 5 years for someone who has just purchased the policy at age 43. The annual effective interest rate is  $i = 6\%$ .

| $x$ | $l_x$    | $d_x$ |
|-----|----------|-------|
| 41  | 10000.00 | 25.66 |
| 42  | 9974.34  | 25.61 |
| 43  | 9948.74  | 25.56 |
| 44  | 9923.17  | 25.52 |
| 45  | 9897.65  | 25.48 |
| 46  | 9872.17  | 25.44 |
| 47  | 9846.74  | 25.40 |
| 48  | 9821.34  | 25.36 |

## 4.6 Variable Insurance Benefits

### Question 30

For the same insurance policy and lifetable as in Question 29, calculate the expected present value of a policy with an initial term of 7 years for an individual who purchased the policy two years ago at age 41. The annual effective interest rate is  $i = 6\%$ .

| $x$ | $l_x$    | $d_x$ |
|-----|----------|-------|
| 41  | 10000.00 | 25.66 |
| 42  | 9974.34  | 25.61 |
| 43  | 9948.74  | 25.56 |
| 44  | 9923.17  | 25.52 |
| 45  | 9897.65  | 25.48 |
| 46  | 9872.17  | 25.44 |
| 47  | 9846.74  | 25.40 |
| 48  | 9821.34  | 25.36 |

## 4.6 Variable Insurance Benefits

### Question 31

A woman aged 30 buys a house with a mortgage of \$200,000. She amortises this amount with annual payments over a period of 8 years at  $i = 6\%$ . She takes out mortgage insurance, which pays off the outstanding balance (principle plus interest) of the mortgage at the end of the year in which she dies. [Assume that the mortgage company does not charge a penalty for early repayment in this case.] If the insurance company uses an interest rate  $i = 5.6\%$  and the life table below, calculate the EPV of the benefit on this policy.

| $x$ | $l_x$    | $d_x$ |
|-----|----------|-------|
| 30  | 10000.00 | 7.25  |
| 31  | 9992.75  | 7.33  |
| 32  | 9985.42  | 7.41  |
| 33  | 9978.00  | 7.50  |
| 34  | 9970.50  | 7.60  |

| $x$ | $l_x$   | $d_x$ |
|-----|---------|-------|
| 35  | 9962.89 | 7.71  |
| 36  | 9955.18 | 7.83  |
| 37  | 9947.35 | 7.96  |
| 38  | 9939.40 | 8.10  |

## 5.4 Annual Life Insurance

### Formulae for Present Value of a Whole-Life Annuity-due

$$\ddot{a}_x = \frac{1 - A_x}{d}$$

$$\ddot{a}_x = \sum_{k=0}^{\infty} v^k {}_k p_x$$

$$\ddot{a}_x = \sum_{k=0}^{\infty} \ddot{a}_{\overline{k+1}|} q_x$$

Variance of present value is given by:

$$\text{Var}(Y) = \frac{{}^2A_x - (A_x)^2}{d^2}$$

### Question 32

Suppose lifetime is modelled as

$$F_0(t) = 1 - \left(1 - \frac{x}{130}\right)^{\frac{1}{4}}$$

Using each of the formulae on the previous slide, calculate the EPV of an annual annuity due on a life currently aged 42, if the annual force of interest is given by  $\delta = 0.04$ .



## 5.4 Annual Life Insurance

### Question 33

A certain insurance company uses the following life table:

| $x$ | $l_x$    | $d_x$ |
|-----|----------|-------|
| 60  | 10000.00 | 24.69 |
| 61  | 9975.31  | 26.76 |
| 62  | 9948.55  | 29.02 |
| 63  | 9919.53  | 31.49 |
| 64  | 9888.04  | 34.17 |
| 65  | 9853.87  | 37.10 |

| $x$ | $l_x$   | $d_x$ |
|-----|---------|-------|
| 66  | 9816.78 | 40.28 |
| 67  | 9776.50 | 43.74 |
| 68  | 9732.76 | 47.50 |
| 69  | 9685.26 | 51.58 |
| 70  | 9633.68 | 56.01 |

From their policy pricing, you determine that they evaluate  $\ddot{a}_{60} = 19.64$  and  $\ddot{a}_{70} = 11.20$ . What rate of interest are they using for these calculations?

(a) 0.687%

(b) 1.223%

(c) 1.891%

(d) 2.433%

## 5.4 Annual Life Insurance

### Question 34

A certain insurance policy involves a term annuity with an annual payment in advance of \$1,200 for a term of 10 years. If the policy is purchased by an individual aged 37, for whom the life table below applies, and the annual interest rate is  $i = 0.065$ , what is the expected present value of this annuity?

| $x$ | $l_x$    | $d_x$ |
|-----|----------|-------|
| 37  | 10000.00 | 4.94  |
| 38  | 9995.06  | 5.17  |
| 39  | 9989.89  | 5.42  |
| 40  | 9984.47  | 5.69  |
| 41  | 9978.77  | 5.99  |
| 42  | 9972.79  | 6.31  |

| $x$ | $l_x$   | $d_x$ |
|-----|---------|-------|
| 43  | 9966.47 | 6.66  |
| 44  | 9959.81 | 7.05  |
| 45  | 9952.76 | 7.47  |
| 46  | 9945.29 | 7.92  |
| 47  | 9937.37 | 8.42  |

## 5.5 Annuities Payable Continuously

### Formulae for Present Value of a Whole-Life Continuous Annuity

$$\bar{a}_x = \frac{1 - \bar{A}_x}{\delta}$$

$$\bar{a}_x = \int_{t=0}^{\infty} e^{-\delta t} {}_t p_x$$

$$\bar{a}_x = \int_{t=0}^{\infty} \bar{a}_{\overline{t}|k} q_x$$

Variance of present value is given by:

$$\text{Var}(Y) = \frac{{}^2\bar{A}_x - (\bar{A}_x)^2}{d^2}$$

## 5.5 Annuities Payable Continuously

### Question 35

Suppose lifetime is modelled as

$$F_0(t) = 1 - \left(1 - \frac{x}{130}\right)^{\frac{1}{4}}$$

Using each of the formulae on the previous slide, calculate the EPV of a continuous annuity at a rate of \$1 per year, on a life currently aged 42, if the annual force of interest is given by  $\delta = 0.04$ .

## 5.6 Annuities Payable 1 / mthly

### Question 36

Using a certain life table, we have that  $A_{65}^{(12)} = 0.5689$ , and the current annual interest rate is  $i^{(12)} = 6\%$ . A man to whom this lifetable applies has saved up \$857,000 for his retirement, and wishes to purchase a whole-life annuity with monthly payments. How much should the monthly payments be?

## 5.7 Comparison of Annuities by Payment Frequency

### Question 37

(a) Order the following from smallest to largest:

- $\bar{a}_x$
- $a_x$
- $\ddot{a}_x$
- $a_x^{(m)}$
- $\ddot{a}_x^{(m)}$

Why are they always in this order?

(b) Does the difference between the different payment frequencies increase or decrease as  $x$  increases?

## 5.8 Deferred Annuities

### Question 38

Mr. Allen is currently aged 49. He has just inherited \$800,000, and he plans to invest it in a deferred annuity, with annual payments starting when he turns 65. If the interest rate is  $i = 7\%$  and the appropriate life table is as given on the next slide, what should the annual payments be in the deferred annuity?

# Life Table for Question 38

| $x$ | $l_x$    | $d_x$  | $x$ | $l_x$   | $d_x$  | $x$ | $l_x$   | $d_x$  |
|-----|----------|--------|-----|---------|--------|-----|---------|--------|
| 49  | 10000.00 | 40.81  | 66  | 8257.75 | 216.46 | 83  | 2190.69 | 385.30 |
| 50  | 9959.19  | 45.19  | 67  | 8041.29 | 235.64 | 84  | 1805.39 | 355.27 |
| 51  | 9914.00  | 50.06  | 68  | 7805.65 | 255.73 | 85  | 1450.12 | 319.28 |
| 52  | 9863.94  | 55.45  | 69  | 7549.92 | 276.57 | 86  | 1130.84 | 278.58 |
| 53  | 9808.49  | 61.42  | 70  | 7273.35 | 297.94 | 87  | 852.26  | 234.91 |
| 54  | 9747.08  | 68.02  | 71  | 6975.42 | 319.54 | 88  | 617.35  | 190.39 |
| 55  | 9679.06  | 75.31  | 72  | 6655.88 | 340.99 | 89  | 426.96  | 147.33 |
| 56  | 9603.75  | 83.34  | 73  | 6314.89 | 361.84 | 90  | 279.63  | 107.97 |
| 57  | 9520.41  | 92.17  | 74  | 5953.05 | 381.53 | 91  | 171.66  | 74.16  |
| 58  | 9428.24  | 101.87 | 75  | 5571.52 | 399.41 | 92  | 97.50   | 47.13  |
| 59  | 9326.37  | 112.50 | 76  | 5172.11 | 414.75 | 93  | 50.37   | 27.24  |
| 60  | 9213.87  | 124.10 | 77  | 4757.36 | 426.75 | 94  | 23.12   | 14.00  |
| 61  | 9089.76  | 136.74 | 78  | 4330.60 | 434.58 | 95  | 9.13    | 6.18   |
| 62  | 8953.02  | 150.46 | 79  | 3896.03 | 437.38 | 96  | 2.95    | 2.23   |
| 63  | 8802.56  | 165.28 | 80  | 3458.64 | 434.39 | 97  | 0.71    | 0.61   |
| 64  | 8637.28  | 181.23 | 81  | 3024.26 | 424.94 | 98  | 0.11    | 0.10   |
| 65  | 8456.05  | 198.30 | 82  | 2599.31 | 408.62 | 99  | 0.01    | 0.01   |



## 5.9 Guaranteed Annuities

### Question 39

An individual aged 65 is planning to purchase a life annuity. She is told that the cost of an annuity that pays \$800 per month is \$120,000. She is also told that the cost for the same annuity for someone 75 years old is \$97,000. If the current rate of interest is  $i^{(12)} = 4\%$  and the individual's probability of surviving to age 75 is 0.87, what is the cost of a life annuity that pays \$800 per month and is guaranteed for 10 years?

## 5.10 Increasing Annuities

### Question 40

A particular disability insurance policy pays \$10000 at the start of the first year of disability, and increases the annual payment by \$1000 every year thereafter for a maximum of 10 years. An individual aged 52 has just become disabled and is starting to claim benefits under this policy. The life table for this individual is as shown below. What is the EPV of the benefits paid out under this policy if the current interest rate is  $i = 6\%$ ?

| $x$ | $l_x$    | $d_x$  |
|-----|----------|--------|
| 52  | 10000.00 | 787.38 |
| 53  | 9212.62  | 811.55 |
| 54  | 8401.07  | 827.99 |
| 55  | 7573.08  | 835.08 |
| 56  | 6738.01  | 831.30 |
| 57  | 5906.71  | 815.36 |

| $x$ | $l_x$   | $d_x$  |
|-----|---------|--------|
| 58  | 5091.35 | 786.36 |
| 59  | 4304.99 | 743.96 |
| 60  | 3561.04 | 688.56 |
| 61  | 2872.48 | 621.47 |
| 62  | 2251.01 | 544.93 |

## 5.10 Increasing Annuities

### Question 41

A woman aged 76 wants to purchase an annual life annuity to fund her retirement. She wants the annuity to pay \$15,000 in the first year and increase by 4% each year. The current interest rate is  $i = 7\%$ , and the life table below is appropriate for this woman. How much should she pay for this annuity?

| $x$ | $l_x$    | $d_x$  | $x$ | $l_x$   | $d_x$  | $x$ | $l_x$  | $d_x$ |
|-----|----------|--------|-----|---------|--------|-----|--------|-------|
| 76  | 10000.00 | 613.73 | 86  | 3394.46 | 584.13 | 96  | 108.49 | 52.47 |
| 77  | 9386.27  | 638.51 | 87  | 2810.32 | 536.22 | 97  | 56.02  | 30.05 |
| 78  | 8747.75  | 659.61 | 88  | 2274.10 | 481.12 | 98  | 25.98  | 15.45 |
| 79  | 8088.14  | 676.05 | 89  | 1792.98 | 420.61 | 99  | 10.53  | 6.94  |
| 80  | 7412.08  | 686.80 | 90  | 1372.36 | 356.98 | 100 | 3.58   | 2.62  |
| 81  | 6725.28  | 690.84 | 91  | 1015.38 | 292.88 | 101 | 0.96   | 0.78  |
| 82  | 6034.45  | 687.22 | 92  | 722.51  | 231.09 | 102 | 0.18   | 0.16  |
| 83  | 5347.23  | 675.13 | 93  | 491.42  | 174.29 | 103 | 0.02   | 0.02  |
| 84  | 4672.10  | 654.02 | 94  | 317.13  | 124.72 |     |        |       |
| 85  | 4018.08  | 623.62 | 95  | 192.41  | 83.91  |     |        |       |

## 5.11 Evaluating Annuity Functions

### Question 42

The price of an annual life annuity-due that pays \$12,000 a year to a life aged 65 is \$112,000. If the current interest rate is  $i^{(12)} = 0.06$ , what is the price for a monthly annuity-due that pays \$1,000 a month to the same life aged 65?

## 5.11 Evaluating Annuity Functions

### Question 43

The EPV of an annual life annuity-due that pays \$20,000 a year to a life aged 65 is \$336,000. The EPV of a monthly annuity-due that pays \$2,000 a month to a life aged 65 is \$392,100. If this is based on the UDD assumption, what rate of interest  $i^{(12)}$  is being applied to calculate the EPV?

- (a) 0.0222      (b) 0.0354      (c) 0.0442      (d) 0.0590

## 5.11 Evaluating Annuity Functions

### Woolhouse's formula

Euler-Maclaurin formula:

$$\int_0^{\infty} g(t) dt = h \sum_{k=0}^{\infty} g(kh) - \frac{h}{2} g(0) + \frac{h^2}{12} g'(0) - \frac{h^4}{720} g'''(0) + \dots$$

Continuous case:

$$\bar{a}_x = \ddot{a}_x - \frac{1}{2} + \frac{1}{12}(\delta + \mu_x)$$

Discrete case:

$$\ddot{a}_x^{(m)} = \ddot{a}_x - \frac{m-1}{2m} - \frac{m^2-1}{12m^2}(\delta + \mu_x)$$

## 5.11 Evaluating Annuity Functions

### Question 44

A certain insurance policy involves a term annuity with a monthly payment in advance of \$100 for a term of 10 years. Suppose the policy is purchased by an individual aged 37, for whom the life table below applies and the current effective interest rate is 6.5%.

| $x$ | $l_x$    | $d_x$ | $x$ | $l_x$   | $d_x$ |
|-----|----------|-------|-----|---------|-------|
| 36  | 10000.00 | 4.73  | 42  | 9968.07 | 6.31  |
| 37  | 9995.27  | 4.94  | 43  | 9961.76 | 6.66  |
| 38  | 9990.33  | 5.17  | 44  | 9955.10 | 7.05  |
| 39  | 9985.16  | 5.42  | 45  | 9948.05 | 7.46  |
| 40  | 9979.74  | 5.69  | 46  | 9940.59 | 7.92  |
| 41  | 9974.05  | 5.99  | 47  | 9932.67 | 8.42  |

Recall from Question 34 that the EPV of an equivalent policy with annual \$1,200 payments is \$9,166.78. Use Woolhouse's formula to calculate the expected present value of this annuity.

## 5.12 Numerical Illustrations

### Question 45

Suppose lifetime is modelled as

$$F_0(t) = 1 - \left(1 - \frac{t}{130}\right)^{\frac{1}{4}}$$

Calculate  $a_x^{(12)}$  for  $i^{(12)} = 0.04$ , for  $x = 20, 30, 40, 50, 60, 70, 80, 90, 100$  using:

- (a) Summing the series explicitly
- (b) Using the UDD assumption
- (c) Using the first 2 terms of Woolhouse's formula
- (d) Using the first 3 terms of Woolhouse's formula
- (e) Using the first 3 terms of Woolhouse's formula with  $\mu_x$  approximated by  $\frac{p_{x-1} + p_{x+1}}{2}$ .



# Solution to Question 45

| $x$ | $A_x$   | Exact   | UDD     | W2      | W3      | W3*     |
|-----|---------|---------|---------|---------|---------|---------|
| 20  | 23.5646 | 23.1040 | 23.1027 | 23.1063 | 23.1028 | 23.1028 |
| 30  | 23.2618 | 22.8017 | 22.7998 | 22.8034 | 22.7999 | 22.7999 |
| 40  | 22.8695 | 22.4104 | 22.4075 | 22.4112 | 22.4076 | 22.4076 |
| 50  | 22.3541 | 21.8965 | 21.8921 | 21.8958 | 21.8922 | 21.8922 |
| 60  | 21.6678 | 21.2123 | 21.2056 | 21.2094 | 21.2058 | 21.2058 |
| 70  | 20.7422 | 20.2902 | 20.2799 | 20.2838 | 20.2802 | 20.2802 |
| 80  | 19.4796 | 19.0333 | 19.0172 | 19.0213 | 19.0176 | 19.0176 |
| 90  | 17.7397 | 17.3024 | 17.2771 | 17.2814 | 17.2776 | 17.2776 |
| 100 | 15.3197 | 14.8971 | 14.8567 | 14.8613 | 14.8573 | 14.8573 |

## 5.13 Functions for Select Lives

### Question 46

The current interest rate is  $i = 0.07$ . The lifetable for select lives is

| $x$ | $l_{[x]}$ | $l_{[x]+1}$ | $l_{[x]+2}$ | $l_{[x]+3}$ |
|-----|-----------|-------------|-------------|-------------|
| 41  | 99824.07  | 99787.27    | 99737.35    | 99669.47    |
| 42  | 99738.86  | 99700.43    | 99648.19    | 99577.01    |
| 43  | 99649.63  | 99609.43    | 99554.64    | 99479.82    |
| 44  | 99556.01  | 99513.84    | 99456.25    | 99377.45    |
| 45  | 99457.54  | 99413.23    | 99352.57    | 99269.38    |
| 46  | 99353.77  | 99307.09    | 99243.07    | 99155.08    |
| 47  | 99244.17  | 99194.90    | 99127.18    | 99033.92    |
| 48  | 99128.18  | 99076.07    | 99004.29    | 98905.24    |
| 49  | 99005.17  | 98949.94    | 98873.71    | 98768.31    |
| 50  | 98874.46  | 98815.81    | 98734.70    | 98622.34    |
| 51  | 98735.31  | 98672.91    | 98586.44    | 98466.45    |

Calculate the EPV of a ten-year life annuity with annual payments of \$30,000, made to an individual aged 42, who was select one year ago.

## 6.4 The Present Value of the Future Loss Random Variable

### Question 47

An insurance policy assumes the following lifetable for an individual currently aged 52:

| $x$ | $l_x$    | $d_x$  |
|-----|----------|--------|
| 52  | 10000.00 | 229.92 |
| 53  | 9770.08  | 262.48 |
| 54  | 9507.60  | 298.55 |
| 55  | 9209.04  | 338.09 |
| 56  | 8870.95  | 380.86 |
| 57  | 8490.09  | 426.34 |

| $x$ | $l_x$   | $d_x$  |
|-----|---------|--------|
| 58  | 8063.75 | 473.71 |
| 59  | 7590.04 | 521.68 |
| 60  | 7068.35 | 568.49 |
| 61  | 6499.86 | 611.79 |
| 62  | 5888.08 | 648.63 |

The policy charges a premium of \$30,000 per year for 10 years, and pays a benefit of \$500,000 in the event of death of the insured. If the interest rate is  $i = 0.04$ , what is the probability that the net future loss on this policy exceeds \$250,000?

## 6.5 The Equivalence Principle

### Question 48

For the policy from Question 47, (see lifetable below) which pays \$500,000 in the event of death of the insured, what annual premium should be charged using the equivalence principle and an interest rate of  $i = 0.04$ ?

| $x$ | $l_x$    | $d_x$  |
|-----|----------|--------|
| 52  | 10000.00 | 229.92 |
| 53  | 9770.08  | 262.48 |
| 54  | 9507.60  | 298.55 |
| 55  | 9209.04  | 338.09 |
| 56  | 8870.95  | 380.86 |
| 57  | 8490.09  | 426.34 |

| $x$ | $l_x$   | $d_x$  |
|-----|---------|--------|
| 58  | 8063.75 | 473.71 |
| 59  | 7590.04 | 521.68 |
| 60  | 7068.35 | 568.49 |
| 61  | 6499.86 | 611.79 |
| 62  | 5888.08 | 648.63 |

## 6.5 The Equivalence Principle

### Question 49

A life aged 55, to whom the lifetable below applies, wants to purchase a deferred annuity that will pay \$24,000 a year, starting in the year when she turns 65. Using the appropriate lifetable, the pensions company determines that the expected payment for an immediate annuity with annual payment \$24,000 for a life aged 65, using the same lifetable, is \$340,000. What annual premium should be charged for the policy if the appropriate interest rate is  $i = 0.04$ ?

| $x$ | $l_x$    | $d_x$ |
|-----|----------|-------|
| 55  | 10000.00 | 9.51  |
| 56  | 9990.49  | 10.00 |
| 57  | 9980.49  | 10.53 |
| 58  | 9969.95  | 11.11 |
| 59  | 9958.85  | 11.73 |
| 60  | 9947.12  | 12.40 |

| $x$ | $l_x$   | $d_x$ |
|-----|---------|-------|
| 61  | 9934.72 | 13.12 |
| 62  | 9921.60 | 13.90 |
| 63  | 9907.70 | 14.74 |
| 64  | 9892.96 | 15.65 |
| 65  | 9877.31 | 16.63 |

## 6.5 The Equivalence Principle

### Question 50

The interest rate is  $i = 0.06$ . A whole life insurance contract for an individual aged 64 with death benefits of \$1,200,000 has annual premiums of \$86,000 payable until age 80. The current interest rate is  $i = 0.05$ . What premium should be charged to a select individual aged 58 for a whole life insurance policy with the same benefit and premiums payable until age 80. Use the table below [assume the table doesn't change over time]. You are also given that the probability of an individual aged 64 surviving to 80 is 0.582314, and that  $A_{80} = 0.7731$ .

| $x$ | $l_{[x]}$ | $l_{[x]+1}$ | $l_{[x]+2}$ | $l_{[x]+3}$ |
|-----|-----------|-------------|-------------|-------------|
| 58  | 99752.79  | 99618.74    | 99442.26    | 99209.76    |
| 59  | 99489.46  | 99344.93    | 99154.47    | 98903.36    |
| 60  | 99205.23  | 99049.28    | 98843.58    | 98572.18    |
| 61  | 98898.23  | 98729.81    | 98507.52    | 98214.03    |
| 62  | 98566.39  | 98384.41    | 98144.05    | 97826.50    |
| 63  | 98207.51  | 98010.76    | 97750.72    | 97407.01    |
| 64  | 97819.18  | 97606.26    | 97324.94    | 96952.78    |

## 6.5 The Equivalence Principle

### Question 51

A life aged 55 wants to purchase a term insurance policy for a term of 10 years. The benefits should be \$800,000. For a policy with annual premiums, the annual premium is \$2,300. The probability that the life dies within the 10 years is 0.0338. If he wants to pay monthly premiums, and benefits to be paid at the end of the month of death, what should the monthly premium be? Assume interest rates are  $i^{(12)} = 0.06$ , and his current mortality is given by  $\mu_{55} = 0.0024$ .

Calculate the premium using:

- (a) Uniform distribution of deaths
- (b) Woolhouse's formula

## 6.6 Gross Premiums

### Question 52

An insurer issues a whole life insurance contract with a death benefit of \$700,000 to a select individual aged 41. Net monthly premiums, payable until age 80 are \$1,380. You are given that  $a_{41:\overline{39}|}^{(12)} = 14.32$ .

The interest rate is  $i^{(12)} = 0.05$ . The insurer has initial costs of \$6,000 plus 50% of the first premium, and renewal costs of 3% of each subsequent premium. What should the gross monthly premium be?



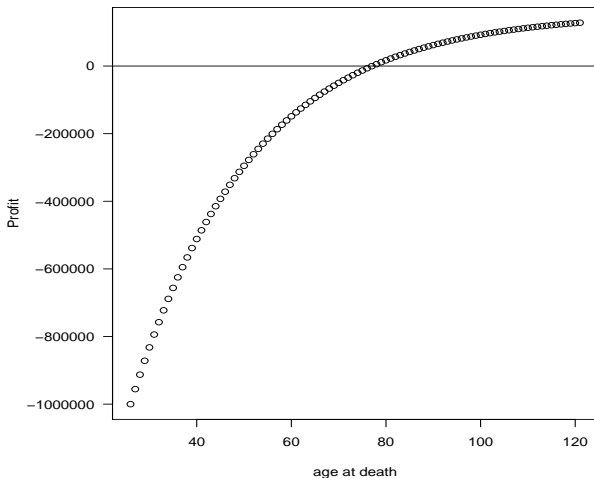
## 6.6 Gross Premiums

### Question 53

A select individual aged 34 wants to purchase endowment insurance for a term of 20 years. The insurance will pay \$600,000 in the event of the individual's death, or at the end of the 20 years. The net premium for this insurance is \$2,300 a month. The current interest rate is  $i^{(12)} = 0.07$ . The insurance company has initial costs of \$1,500 plus 70% of the first premium and monthly renewal costs of 0.01% of the insured benefit. Calculate the gross premium.

## 6.7 Profit

**Figure:** Net profit at time of starting policy on a whole life insurance policy with death benefit \$1,000,000, annual premium \$6,000, interest rate  $i = 0.04$ , as a function of age at death.



## 6.7 Profit

### Question 54

A whole life insurance policy to an individual aged 54 has a death benefit of \$700,000 and an annual premium of \$28,000. If interest is  $i = 0.07$  and the life table for this life is as below, what is the probability that the policy makes a net profit?

| $x$ | $l_x$    | $d_x$ |
|-----|----------|-------|
| 54  | 10000.00 | 18.07 |
| 55  | 9981.93  | 19.80 |
| 56  | 9962.13  | 21.70 |
| 57  | 9940.43  | 23.77 |
| 58  | 9916.66  | 26.04 |
| 59  | 9890.63  | 28.52 |
| 60  | 9862.11  | 31.23 |
| 61  | 9830.88  | 34.19 |

| $x$ | $l_x$   | $d_x$ |
|-----|---------|-------|
| 62  | 9796.70 | 37.42 |
| 63  | 9759.28 | 40.94 |
| 64  | 9718.33 | 44.79 |
| 65  | 9673.55 | 48.97 |
| 66  | 9624.58 | 53.53 |
| 67  | 9571.05 | 58.48 |
| 68  | 9512.57 | 63.85 |
| 69  | 9448.72 | 69.68 |

## 6.8 The Portfolio Percentile Premium Principle

### Question 55

An insurer issues 10-year term insurance to 200 lives aged 43. The policies have a death benefit of \$1,000,000. The initial costs are \$1,000 plus 40% of the first premium. The renewal costs are 1.5% of each subsequent premium. The following life table is to be used:

| $x$ | $l_x$    | $d_x$ |
|-----|----------|-------|
| 43  | 10000.00 | 8.64  |
| 44  | 9991.36  | 9.55  |
| 45  | 9981.80  | 10.58 |
| 46  | 9971.23  | 11.73 |
| 47  | 9959.50  | 13.01 |
| 48  | 9946.49  | 14.45 |

| $x$ | $l_x$   | $d_x$ |
|-----|---------|-------|
| 49  | 9932.04 | 16.07 |
| 50  | 9915.97 | 17.87 |
| 51  | 9898.10 | 19.89 |
| 52  | 9878.20 | 22.16 |
| 53  | 9856.04 | 24.69 |

If interest is  $i^{(12)} = 0.06$  and the UDD assumption is to be used, calculate the premium which ensures a 95% probability that the policies will result in a profit.

## 6.9 Extra Risks

### Question 56

The interest rate is 0.04. The standard lifetable for women is:

| $x$ | $l_x$    | $d_x$ | $x$ | $l_x$   | $d_x$ | $x$ | $l_x$   | $d_x$ |
|-----|----------|-------|-----|---------|-------|-----|---------|-------|
| 36  | 10000.00 | 2.76  | 42  | 9979.48 | 4.52  | 48  | 9945.73 | 7.47  |
| 37  | 9997.24  | 3.00  | 43  | 9974.96 | 4.91  | 49  | 9938.26 | 8.13  |
| 38  | 9994.24  | 3.25  | 44  | 9970.05 | 5.34  | 50  | 9930.13 | 8.85  |
| 39  | 9990.99  | 3.53  | 45  | 9964.72 | 5.80  | 51  | 9921.27 | 9.64  |
| 40  | 9987.47  | 3.83  | 46  | 9958.91 | 6.31  | 52  | 9911.63 | 10.49 |
| 41  | 9983.64  | 4.16  | 47  | 9952.60 | 6.87  | 53  | 9901.14 | 11.43 |

Calculate the annual premium for a 10-year term insurance with death benefit \$1,000,000 on:

- a standard life aged 36.
- an impaired life aged 36, who is treated as being 5 years older.
- a life aged 36, who works in a hazardous environment which increases her mortality rate by 0.006.
- a life aged 36 whose mortality is 1.2 times the normal mortality.

## 6.9 Extra Risks

**Table:** Annual Premiums for whole life insurance with death benefits \$100,000, premiums payable until death, interest rate  $i = 0.04$ , mortality following Makeham's model with  $A = 0.0000708$ ,  $B = 0.00001044$ ,  $C = 1.121$ , under various adjustments for extra risks.

| age | Standard Premium | age+5    | mortality+0.006 | mortality $\times$ 1.2 |
|-----|------------------|----------|-----------------|------------------------|
| 30  | 771.10           | 969.68   | 1215.18         | 829.91                 |
| 40  | 1228.44          | 1569.95  | 1651.40         | 1328.00                |
| 50  | 2027.53          | 2651.41  | 2430.55         | 2206.41                |
| 60  | 3519.47          | 4755.59  | 3906.35         | 3867.22                |
| 70  | 6562.38          | 9280.27  | 6938.24         | 7310.16                |
| 80  | 13495.27         | 20236.19 | 13858.15        | 15306.74               |
| 90  | 31334.00         | 50068.37 | 31642.40        | 36271.61               |
| 100 | 81856.71         | —        | 81931.27        | 96153.85               |

## 7.3 Policies with Annual Cash Flows

### Question 57

The life table for standard lives is

| $x$ | $l_x$    | $d_x$ |
|-----|----------|-------|
| 37  | 10000.00 | 4.52  |
| 38  | 9995.48  | 4.90  |
| 39  | 9990.58  | 5.32  |
| 40  | 9985.25  | 5.78  |
| 41  | 9979.47  | 6.29  |
| 42  | 9973.18  | 6.85  |

| $x$ | $l_x$   | $d_x$ |
|-----|---------|-------|
| 43  | 9966.33 | 7.46  |
| 44  | 9958.86 | 8.14  |
| 45  | 9950.72 | 8.88  |
| 46  | 9941.84 | 9.70  |
| 47  | 9932.14 | 10.60 |

The interest rate is  $i = 0.08$ . The resulting annual net premium for a 10-year term insurance with a death benefit of \$200,000 on a life aged 37 is therefore \$119.32. Calculate the policy value at each year of the term of the policy.

## 7.3 Policies with Annual Cash Flows

**Table:** Policy values for a 10-year life insurance policy on a life aged 37 for each year of the policy

| age | $a_x$ | $A_x$  | Premium | Policy Value |
|-----|-------|--------|---------|--------------|
| 37  | 7.23  | 862.82 | 119.32  | 0.00         |
| 38  | 6.73  | 841.83 | 125.04  | 38.49        |
| 39  | 6.19  | 811.53 | 131.01  | 72.42        |
| 40  | 5.61  | 770.36 | 137.25  | 100.64       |
| 41  | 4.98  | 716.64 | 143.77  | 121.85       |
| 42  | 4.31  | 648.32 | 150.56  | 134.50       |
| 43  | 3.57  | 563.20 | 157.62  | 136.85       |
| 44  | 2.78  | 458.90 | 165.01  | 127.06       |
| 45  | 1.93  | 332.41 | 172.67  | 102.70       |
| 46  | 1.00  | 180.68 | 180.68  | 61.36        |



## 7.3 Policies with Annual Cash Flows

### Comments on Policy Values

- Policy value basis may have changed since the policy was introduced.
- Net premium policy value uses a premium based on the new policy value basis.
- Gross premium policy value uses actual contract premium.
- If gross policy value at time 0 is positive, it means the valuation basis is more conservative than the basis used to calculate the premium. If it is negative, it means the premium basis is more conservative.
- Note how the policy value increases over time. The early premiums more than cover the risks in that year. The surplus is necessary to cover expected losses in future years.

## 7.3 Policies with Annual Cash Flows

### Question 58

Repeat Question 57 using the select life table below. [Assume the life is select at 37].

| $x$ | $l_{[x]}$ | $l_{[x]+1}$ | $l_{[x]+2}$ | $l_{[x]+3}$ |
|-----|-----------|-------------|-------------|-------------|
| 37  | 9995.86   | 9993.18     | 9989.73     | 9985.25     |
| 38  | 9990.99   | 9988.08     | 9984.33     | 9979.47     |
| 39  | 9985.70   | 9982.54     | 9978.46     | 9973.18     |
| 40  | 9979.95   | 9976.52     | 9972.08     | 9966.33     |
| 41  | 9973.69   | 9969.97     | 9965.13     | 9958.86     |
| 42  | 9966.88   | 9962.83     | 9957.56     | 9950.72     |
| 43  | 9959.46   | 9955.04     | 9949.30     | 9941.84     |
| 44  | 9951.37   | 9946.55     | 9940.29     | 9932.14     |
| 45  | 9942.55   | 9937.29     | 9930.45     | 9921.55     |
| 46  | 9932.91   | 9927.16     | 9919.69     | 9909.97     |
| 47  | 9922.38   | 9916.10     | 9907.94     | 9897.30     |

## 7.3 Policies with Annual Cash Flows

**Table:** Policy values for a 10-year life insurance policy on a select life aged 37 for each year of the policy

| age | $a_x$ | $A_x$  | Premium | Policy Value |
|-----|-------|--------|---------|--------------|
| 37  | 7.23  | 790.99 | 109.35  | 0.00         |
| 38  | 6.73  | 800.87 | 118.93  | 64.50        |
| 39  | 6.19  | 796.16 | 128.52  | 118.75       |
| 40  | 5.61  | 770.51 | 137.28  | 156.73       |
| 41  | 4.98  | 716.79 | 143.80  | 171.70       |
| 42  | 4.31  | 648.49 | 150.60  | 177.59       |
| 43  | 3.57  | 563.39 | 157.67  | 172.65       |
| 44  | 2.78  | 458.90 | 165.01  | 154.78       |
| 45  | 1.93  | 332.41 | 172.67  | 121.89       |
| 46  | 1.00  | 180.68 | 180.68  | 71.33        |

## 7.3 Policies with Annual Cash Flows

### Question 59

A life aged 44 purchases a 10-year endowment policy, which pays a benefit of \$1,000,000 either at the end of 10 years, or at the end of the year of death. The appropriate life table is given below. The annual premiums are calculated as \$72,111.08.

| $x$ | $l_x$    | $d_x$ |
|-----|----------|-------|
| 44  | 10000.00 | 8.08  |
| 45  | 9991.92  | 8.84  |
| 46  | 9983.08  | 9.68  |
| 47  | 9973.39  | 10.61 |
| 48  | 9962.78  | 11.64 |
| 49  | 9951.14  | 12.77 |

| $x$ | $l_x$   | $d_x$ |
|-----|---------|-------|
| 50  | 9938.37 | 14.02 |
| 51  | 9924.35 | 15.40 |
| 52  | 9908.95 | 16.92 |
| 53  | 9892.03 | 18.59 |
| 54  | 9873.44 | 20.44 |

Using a policy value basis of interest rate 0.05%:

- calculate the gross policy value at time 0.
- calculate the gross and net policy values at time 5.

## 7.3 Policies with Annual Cash Flows

### Question 60

An insurance policy is paid to a life aged 55. The life follows the life table below.

| $x$ | $l_x$    | $d_x$ |
|-----|----------|-------|
| 55  | 10000.00 | 23.94 |
| 56  | 9976.06  | 26.35 |
| 57  | 9949.71  | 29.01 |
| 58  | 9920.70  | 31.93 |
| 59  | 9888.77  | 35.15 |
| 60  | 9853.62  | 38.68 |

| $x$ | $l_x$   | $d_x$ |
|-----|---------|-------|
| 61  | 9814.94 | 42.57 |
| 62  | 9772.38 | 46.83 |
| 63  | 9725.54 | 51.51 |
| 64  | 9674.04 | 56.63 |
| 65  | 9617.40 | 62.24 |

The policy pays a benefit of \$600,000 if the life survives to age 65. If the life dies before age 65, at the end of the year of death, the policy pays a benefit equal to the policy value at the start of the year in which the life dies. If the annual premium is \$54,000 and the interest rate is  $i = 0.07$ , calculate the policy value after 3 years.

## 7.3 Policies with Annual Cash Flows

### Question 61

Recall Question 57, where the annual net premium for a 10-year term insurance with a death benefit of \$200,000 on a life aged 37 was calculated as \$119.32. The policy value after 1 year was calculated as \$38.49, based on the lifetable with  $q_{37} = 0.000452$  and an interest rate  $i = 0.08$ .

Suppose the company sells 1,200 such policies in a given year, and in the first year of the policies:

- one policy holder dies
- the company earns an interest rate of  $i = 0.095$
- the company needs to pay expenses of \$1,000 related to the policies at the end of the year.

What is the company's profit or loss for the year on these policies?

## 7.3 Policies with Annual Cash Flows

### Question 62

Recall Question 57, where the annual net premium for a 10-year term insurance with a death benefit of \$200,000 on a life aged 37 was calculated as \$119.32.

Suppose the company sells 1,200 such policies in a given year, and in the first three years of the policies:

- one policyholder dies in the first year.
- the company earns an annual interest rate of  $i = 0.085$  in the first year, and  $i = 0.09$  over the remaining two years.
- the company needs to pay expenses related to the policies of \$1,000 at the end of each year.

Calculate the asset share of the remaining policies.

## 7.4 Policies with 1/*m*thly Cash Flows

### Question 63

As for Question 59,  $i = 0.05$  and the lifetable for a life aged 44 is

| $x$ | $l_x$    | $d_x$ |
|-----|----------|-------|
| 44  | 10000.00 | 8.08  |
| 45  | 9991.92  | 8.84  |
| 46  | 9983.08  | 9.68  |
| 47  | 9973.39  | 10.61 |
| 48  | 9962.78  | 11.64 |
| 49  | 9951.14  | 12.77 |

| $x$ | $l_x$   | $d_x$ |
|-----|---------|-------|
| 50  | 9938.37 | 14.02 |
| 51  | 9924.35 | 15.40 |
| 52  | 9908.95 | 16.92 |
| 53  | 9892.03 | 18.59 |
| 54  | 9873.44 | 20.44 |

A life aged 44 purchases a 10-year endowment policy, which pays a benefit of \$1,000,000 either at the end of 10 years, or at the end of the month of death. For annual premiums, we have  $A_{44:\overline{10}|} = 0.61556$  and  $A_{49:\overline{5}|} = 0.784108$ . Using Woolhouse's formula:

- calculate the monthly premium.
- calculate the policy value after 5 years 2 months.



## 7.4 Policies with 1/*m*thly Cash Flows

### Question 64

As for Question 59,  $i = 0.05$  and the lifetable for a life aged 44 is

| $x$ | $l_x$    | $d_x$ |
|-----|----------|-------|
| 44  | 10000.00 | 8.08  |
| 45  | 9991.92  | 8.84  |
| 46  | 9983.08  | 9.68  |
| 47  | 9973.39  | 10.61 |
| 48  | 9962.78  | 11.64 |
| 49  | 9951.14  | 12.77 |

| $x$ | $l_x$   | $d_x$ |
|-----|---------|-------|
| 50  | 9938.37 | 14.02 |
| 51  | 9924.35 | 15.40 |
| 52  | 9908.95 | 16.92 |
| 53  | 9892.03 | 18.59 |
| 54  | 9873.44 | 20.44 |

A life aged 44 purchases a 10-year endowment policy, which pays a benefit of \$1,000,000 either at the end of 10 years, or at the end of the month of death. Recall from the previous question that the policy value after 5 years 2 months is \$482,621.83.

Calculate the policy value after 5 years 1.6 months.

## 7.5 Policies with Continuous Cash Flows

### Thiele's differential equation

$$\frac{d}{dt} {}_tV = \delta_t {}_tV + P_t - (S_t - {}_tV)\mu_{x+t}$$

where:

- ${}_tV$  is the policy value at time  $t$  if the policy is still in force.
- $\delta_t$  is the force of interest at time  $t$ .
- $P_t$  is the rate of premium payment at time  $t$  (minus expenses).
- $S_t$  is the death benefit at time  $t$  (plus expenses).
- $\mu_{x+t}$  is the force of mortality at time  $t$ .

## 7.5 Policies with Continuous Cash Flows

### Question 65

Suppose Mortality is given by Makeham's law

$$\mu_x = 0.000084 + 0.0000104(1.099)^x$$

Use a numerical solution to Thiele's differential equation to calculate the policy value of a 10-year term insurance with death benefit \$500,000 and force of interest given by  $\delta = 0.045$ , sold to a life aged 45, with the appropriate net premium (paid continuously) for this policy.

## 7.5 Policies with Continuous Cash Flows

### Answer to Question 65

| $h$      | 0.1     | 0.01   | 0.001  | 0.0001 | 0.00005 |
|----------|---------|--------|--------|--------|---------|
| Premium= | 569.63  | 620.10 | 623.34 | 625.66 | 625.69  |
| age=45   | 0       | 0      | 0      | 0      | 0       |
| age=46   | 151.09  | 201.40 | 204.59 | 206.95 | 206.98  |
| age=47   | 270.91  | 373.65 | 380.16 | 385.00 | 385.05  |
| age=48   | 354.20  | 511.59 | 521.55 | 528.98 | 529.07  |
| age=49   | 395.09  | 609.44 | 622.99 | 633.12 | 633.24  |
| age=50   | 387.01  | 660.72 | 677.99 | 690.95 | 691.11  |
| age=51   | 322.62  | 658.15 | 679.29 | 695.22 | 695.41  |
| age=52   | 193.69  | 593.62 | 618.78 | 637.81 | 638.04  |
| age=53   | -8.96   | 458.05 | 487.37 | 509.65 | 509.91  |
| age=54   | -295.58 | 241.28 | 274.92 | 300.60 | 300.90  |
| age=55   | -677.63 | -68.05 | -29.93 | -0.68  | -0.34   |

## 7.6 Policy Alterations

### Common Policy Alterations

- Surrender – cancel policy with immediate effect, and may receive a surrender value.
- Stop paying premiums and receive a reduced sum insured.
- Convert from whole-life to paid-up term

### Reasons for Penalising Alterations

- Adverse selection
- Expenses in changing policy
- Liquidity risk

## 7.6 Policy Alterations

### Question 66

Six years ago a woman then aged 37 purchased whole life insurance for a death benefit of \$400,000. The appropriate life table is

| $x$ | $l_x$    | $d_x$ | $x$ | $l_x$   | $d_x$ | $x$ | $l_x$   | $d_x$ |
|-----|----------|-------|-----|---------|-------|-----|---------|-------|
| 37  | 10000.00 | 4.11  | 43  | 9969.60 | 6.70  | 49  | 9919.02 | 11.37 |
| 38  | 9995.89  | 4.44  | 44  | 9962.89 | 7.31  | 50  | 9907.65 | 12.45 |
| 39  | 9991.45  | 4.81  | 45  | 9955.59 | 7.97  | 51  | 9895.21 | 13.63 |
| 40  | 9986.64  | 5.22  | 46  | 9947.62 | 8.70  | 52  | 9881.57 | 14.94 |
| 41  | 9981.42  | 5.67  | 47  | 9938.92 | 9.50  | 53  | 9866.63 | 16.38 |
| 42  | 9975.76  | 6.16  | 48  | 9929.42 | 10.39 | 54  | 9850.25 | 17.96 |

The annual premium is \$2002.57 ( $i = 0.05$ ). The policy has a cash surrender value of 85% of the policy value minus \$150.

- Calculate the cash surrender value of the policy.
- If the woman wishes to change the insurance to endowment insurance, which will pay a benefit of \$100,000 either when she turns 55 or at death, what should the new premiums be?

## 7.6 Policy Alterations

### Question 67

Four years ago, a man aged 53 took out a term insurance policy with a term of 10 years and a death benefit of \$250,000. The insurance company estimated that the probability of his dying during the term of the insurance was 0.021, and was calculated the monthly net premium as \$54.71. If the man wanted to take out the same term insurance today for the remaining 6 years, the monthly premium would be \$61.17, and his chance of dying during the term would be 0.018. The interest rate is  $i = 0.04$ . The policy has a cash surrender value of 85% of the policy value.

- If the man wants to convert the policy into a paid-up policy for the remainder of the term (i.e. the next six years), what are the new death benefits?
- If the man wants to reduce his monthly premiums to \$25, what are the new death benefits?

## 7.7 Retrospective Policy Values

### Question 68

A man aged 44 buys a whole life insurance policy. The annual premiums are \$600 for the first 4 years, and \$1,000 after then until age 80. The death benefits are \$80,000 for the first 3 years, \$100,000 for the next 4 years and \$400,000 from then on. The interest rate is  $i = 0.07$  and the appropriate lifetable is:

| $x$ | $l_x$    | $d_x$ |
|-----|----------|-------|
| 44  | 10000.00 | 3.45  |
| 45  | 9996.55  | 3.72  |
| 46  | 9992.83  | 4.02  |
| 47  | 9988.82  | 4.35  |

| $x$ | $l_x$   | $d_x$ |
|-----|---------|-------|
| 48  | 9984.47 | 4.71  |
| 49  | 9979.75 | 5.12  |
| 50  | 9974.64 | 5.57  |
| 51  | 9969.07 | 6.06  |

$$A_{(44)} = 0.0403706 \quad A_{(46)} = 0.0455118 \quad A_{(47)} = 0.0483148$$

$$A_{(52)} = 0.0649932 \quad A_{(51)} = 0.0612725 \quad A_{(50)} = 0.057754$$

$$A_{(49)} = 0.0544274 \quad A_{(48)} = 0.0512836 \quad A_{(80)} = 0.292719$$

Calculate the retrospective policy value after 2 years.



## 7.9 Deferred Acquisition Expenses and Modified Premium Reserves Payments

### Question 69

A man aged 29 buys a whole-life insurance policy with death benefits of \$300,000. The lifetable for this man is:

| $x$ | $l_x$    | $d_x$ |
|-----|----------|-------|
| 29  | 10000.00 | 2.28  |
| 30  | 9997.72  | 2.40  |
| 31  | 9995.33  | 2.53  |
| 32  | 9992.80  | 2.68  |

The interest rate is  $i = 0.06$ .

You are given that  $A_{(29)} = 0.0343763$ ,  $A_{(30)} = 0.0362191$ ,  $A_{(31)} = 0.0381614$ ,  $A_{(32)} = 0.0402081$ . Using the lifetable above, and using a Full Preliminary Term of 1 year, calculate the policy value after 2 years.