ACSC/STAT 3720, Life Contingencies I
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Homework Sheet 1
Model Solutions

Basic Questions

1. *An insurance company models the future lifetime of an individual as having survival function $S(x) = e^{-\left(\frac{x}{75}\right)^3}$. Calculate the force of mortality.*

   The density function is $f(x) = -S'(x) = 3\frac{x^2}{75^3}e^{-\left(\frac{x}{75}\right)^3}$. The force of mortality is $\frac{f(x)}{S(x)} = \frac{3\frac{x^2}{75^3}e^{-\left(\frac{x}{75}\right)^3}}{e^{-\left(\frac{x}{75}\right)^3}} = 3\frac{x^2}{75^3}$.

2. *An insurance company models the future lifetime of an individual as having survival function $S(x) = e^{-\frac{x}{72}}$. Calculate:*

   (a) The mean and standard deviation of T_x.

   The distribution of the remaining future lifetime is exponential with mean 72, by the memoryless property of the exponential distribution. The mean is therefore 72 and the standard deviation is also 72.

   (b) The mean curtate future lifetime.

   The mean curtate future lifetime is given by

$$\begin{aligned}
\sum_{n=0}^{\infty} n\left(e^{-\frac{n}{72}} - e^{-\frac{n+1}{72}}\right) &= \sum_{n=0}^{\infty} ne^{-\frac{n}{72}} - \sum_{m=1}^{\infty}(m-1)e^{-\frac{m}{72}} \\
&= \sum_{n=1}^{\infty}\left(ne^{-\frac{n}{72}} - (n-1)e^{-\frac{n}{72}}\right) \\
&= \sum_{n=1}^{\infty} e^{-\frac{n}{72}} \\
&= \frac{1}{1-e^{-\frac{1}{72}}} - 1 \\
&= 71.50116
\end{aligned}$$

3. *An insurance company uses a survival model with survival function ${}_tp_x = \left(1 - \frac{t}{\omega - x}\right)^{\frac{1}{6}}$, where ω is the maximum age attainable. The company wants to ensure that the life expectancy of an individual aged 60 under this model is 20 years. What age should they choose as the maximum age attainable?*

Life expectancy from age 60 is given by

$$\begin{aligned}
\int_0^{\omega-60} {}_tp_{60}\,dt &= \int_0^{\omega-60} \left(1-\frac{t}{\omega-60}\right)^{\frac{1}{6}}\,dt \\
&= \int_0^1 (\omega-60)u^{\frac{1}{6}}\,du \\
&= \left[\frac{6(\omega-60)}{7}u^{\frac{7}{6}}\right]_0^1 \\
&= \frac{6(\omega-60)}{7}
\end{aligned}$$

Setting this equal to 20 gives

$$\begin{aligned}
\frac{6(\omega-60)}{7} &= 20 \\
6(\omega-60) &= 140 \\
\omega-60 &= \frac{140}{6} \\
\omega &= \frac{140}{6}+60 = 83.333
\end{aligned}$$

4. *An insurance company uses a survival model given by*

$$S_0(x) = \frac{1}{2}\left(1-\frac{x}{95}\right)^{\frac{1}{6}} + \frac{1}{2}\left(1-\frac{x}{115}\right)^{\frac{1}{4}}$$

Using this model, prepare a life table for the ages from 40 to 45, using radix 10,000.

We calculate

$$S_0(40) = \frac{1}{2}\left(1 - \frac{40}{95}\right)^{\frac{1}{6}} + \frac{1}{2}\left(1 - \frac{40}{115}\right)^{\frac{1}{4}} = 0.9057928$$
$$S_0(41) = \frac{1}{2}\left(1 - \frac{41}{95}\right)^{\frac{1}{6}} + \frac{1}{2}\left(1 - \frac{41}{115}\right)^{\frac{1}{4}} = 0.9028936$$
$$S_0(42) = \frac{1}{2}\left(1 - \frac{42}{95}\right)^{\frac{1}{6}} + \frac{1}{2}\left(1 - \frac{42}{115}\right)^{\frac{1}{4}} = 0.8999575$$
$$S_0(43) = \frac{1}{2}\left(1 - \frac{43}{95}\right)^{\frac{1}{6}} + \frac{1}{2}\left(1 - \frac{43}{115}\right)^{\frac{1}{4}} = 0.8969832$$
$$S_0(44) = \frac{1}{2}\left(1 - \frac{44}{95}\right)^{\frac{1}{6}} + \frac{1}{2}\left(1 - \frac{44}{115}\right)^{\frac{1}{4}} = 0.8939696$$
$$S_0(45) = \frac{1}{2}\left(1 - \frac{45}{95}\right)^{\frac{1}{6}} + \frac{1}{2}\left(1 - \frac{45}{115}\right)^{\frac{1}{4}} = 0.8909155$$

This gives

$$S_{40}(40) = 1.0000000$$
$$S_{40}(41) = 0.9967993$$
$$S_{40}(42) = 0.9935578$$
$$S_{40}(43) = 0.9902742$$
$$S_{40}(44) = 0.9869471$$
$$S_{40}(45) = 0.9835753$$

The lifetable is therefore

x	l_x	d_x
40	10000.00	32.01
41	9967.99	32.42
42	9935.58	32.84
43	9902.74	33.27
44	9869.47	33.72
45	9835.75	34.18

5. *Using the lifetable:*

x	l_x	d_x
50	*10000.00*	*6.44*
51	*9993.56*	*7.05*
52	*9986.51*	*7.73*
53	*9978.78*	*8.48*
54	*9970.30*	*9.31*
55	*9960.99*	*10.24*
56	*9950.75*	*11.27*
57	*9939.48*	*12.42*

calculate the probability that an individual aged 50 years and four months survives another 5 years, using:

(a) the uniform distribution of deaths assumption.

Using UDD, $l_{50\frac{1}{3}} = \frac{1}{3} \times 9993.56 + \frac{2}{3} \times 10000.00 = 9997.853333$ and $l_{55\frac{1}{3}} = \frac{1}{3} \times 9950.75 + \frac{2}{3} \times 9960.99 = 9957.576667$.

The probability of surviving is therefore $\frac{9957.576667}{9997.853333} = 0.9959715$.

(b) the constant force of mortality assumption.

Using constant force of mortality, we get $\mu_x = -\log(0.999356)$ for $50 < x < 51$, so $l_{50\frac{1}{3}} = 10000e^{-\frac{\log(0.999356)}{3}} = 9997.852872$ and $\mu_x = -\log\left(\frac{9950.75}{9960.99}\right)$ for $55 < x < 56$, so $l_{55\frac{1}{3}} = 9960.99e^{-\frac{\log\left(\frac{9950.75}{9960.99}\right)}{3}} = 9957.575496$. The probability of surviving 5 years is therefore $\frac{9957.575496}{9997.852872} = 0.9959714$.

Standard Questions

6. *An insurance company wants to use a model of mortality of the form $\mu_x = \frac{a}{100-x} + \frac{b}{120-x}$. Based on the company's data, an individual aged 65 has probability 0.868 of surviving to age 80, and the probability of an individual aged 45 surviving to age 65 is 0.890. It is extremely important for these properties to match the observed data. The company chooses the values of a and b to ensure that these observations are matched by the model. What values of a and b should they choose?*

The model gives $S_x(t) = e^{-\int_0^t \mu_{x+s}\,ds}$. We have

$$\int_0^t \frac{a}{100-(x+s)} + \frac{b}{120-(x+s)}\,ds = [-a\log(100-x-s) - b\log(120-x-s)]_0^t = a\log\left(\frac{100-x}{100-x-t}\right) + b\log$$

We therefore get $S_x(t) = \left(\frac{100-x-t}{100-x}\right)^a \left(\frac{120-x-t}{120-x}\right)^b$.

The data we need to match is therefore

$$\left(\frac{20}{35}\right)^a\left(\frac{40}{55}\right)^b = 0.868$$
$$\left(\frac{35}{55}\right)^a\left(\frac{55}{75}\right)^b = 0.890$$

taking logs gives:

$$\begin{aligned}
a\log\left(\frac{20}{35}\right)+b\log\left(\frac{40}{55}\right) &= \log(0.868)\\
a\log\left(\frac{35}{55}\right)+b\log\left(\frac{55}{75}\right) &= \log(0.890)\\
0.05294926655b &= 0.002197111472\\
b &= 0.04149465356\\
a &= 0.2293527807
\end{aligned}$$

7. *An insurance company prepares the following lifetable for an individual.*

x	l_x	d_x
35	*10000.00*	*71.39*
36	*9928.61*	*73.74*
37	*9854.87*	*76.47*
38	*9778.40*	*79.60*
39	*9698.80*	*83.20*
40	*9615.59*	*87.33*

After an examination, it determines that the individual's probability of death at age 35 is actually 0.023. The probability of death in each subsequent year remains the same. Prepare a new life table for this individual over the same range using radix 10,000.

From the table, we calculate

x	q_x
36	$\frac{73.74}{9928.61} = 0.007427021507$
37	$\frac{76.47}{9854.87} = 0.007759615297$
38	$\frac{79.60}{9778.40} = 0.008140391066$
39	$\frac{83.20}{9698.80} = 0.008578380831$
40	$\frac{87.33}{9615.59} = 0.009082126006$

This gives the lifetable

x	l_x	d_x
35	10000.00	230.00
36	9770.00	72.56
37	9697.44	75.25
38	9622.19	78.33
39	9543.86	81.87
40	9461.99	85.93