

ACSC/STAT 3720, Life Contingencies I  
Winter 2017  
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Homework Sheet 1  
Model Solutions

### Basic Questions

1. An insurance company models the future lifetime of an individual as having survival function  $S(x) = e^{-\left(\frac{x}{90}\right)^4}$ . Calculate the force of mortality.

The density function is  $f(x) = -\frac{dS(x)}{dx} = \frac{4}{90} \left(\frac{x}{90}\right)^3 e^{-\left(\frac{x}{90}\right)^4}$ , so force of mortality is

$$\mu(x) = \frac{\frac{4}{90} \left(\frac{x}{90}\right)^3 e^{-\left(\frac{x}{90}\right)^4}}{e^{-\left(\frac{x}{90}\right)^4}} = \frac{4}{90} \left(\frac{x}{90}\right)^3$$

Alternatively, we can use

$$\mu(x) = \frac{d}{dx}(-\log(S(x))) = \frac{d}{dx} \left( \left(\frac{x}{90}\right)^4 \right)$$

2. An insurance company models the future lifetime of an individual as having survival function  $S(x) = e^{-\frac{x}{63}}$ . Calculate:

(a) The mean and standard deviation of  $T_x$ .

The mean of  $T_x$  is given by

$$\int_0^\infty \frac{S(x+t)}{S(x)} dt = \int_0^\infty e^{-\frac{t}{63}} dx = \left[ -63e^{-\frac{t}{63}} \right]_0^\infty = 63$$

The mean of  $T_x^2$  is given by

$$\int_0^\infty 2t \frac{S(x+t)}{S(x)} dt = \int_0^\infty 2te^{-\frac{t}{63}} dx = \left[ -126te^{-\frac{t}{63}} \right]_0^\infty + \int_0^\infty 126e^{-\frac{t}{63}} dx = 126*63$$

so the variance of  $T_x$  is  $126 * 63 - 63^2 = 63^2$ , and the standard deviation is 63.

We could also get this directly by observing that  $T_x$  follows an exponential distribution with mean 63.

(b) The mean curtate future lifetime.

For the mean curtate future lifetime, the probability of surviving  $n$  years is  $e^{-\frac{n}{63}}$ , so the mean curtate future lifetime is

$$\sum_{n=0}^{\infty} e^{-\frac{n}{63}} = \frac{1}{1 - e^{-\frac{1}{63}}} - 1 = 62.50132$$

3. An insurance company uses a survival model with survival function  ${}_t p_x = \left(1 - \frac{t}{130-x}\right)^\alpha$ . The company wants to ensure that the life expectancy of an individual aged 60 under this model is 20 years. What value of  $\alpha$  should they choose?

The life expectancy of an individual aged 60 is given by

$$\int_0^{70} \left(1 - \frac{t}{70}\right)^\alpha dt$$

Substituting  $u = 1 - \frac{t}{70}$  this becomes

$$70 \int_0^1 u^\alpha du = 70 \left[ \frac{u^{\alpha+1}}{\alpha+1} \right]_0^1 = \frac{70}{\alpha+1}$$

We therefore have

$$\begin{aligned} \frac{70}{\alpha+1} &= 20 \\ \alpha+1 &= \frac{7}{2} \\ \alpha &= 2.5 \end{aligned}$$

4. An insurance company uses a survival model given by

$$S_0(x) = \frac{1}{2} \left(1 - \frac{x}{95}\right)^{\frac{1}{6}} + \frac{1}{2} \left(1 - \frac{x}{125}\right)^{\frac{1}{3}}$$

Using this model, prepare a life table for the ages from 40 to 45, using radix 10,000.

We first calculate

$${}_t p_{40} = \frac{\frac{1}{2} \left(1 - \frac{40+t}{95}\right)^{\frac{1}{6}} + \frac{1}{2} \left(1 - \frac{40+t}{125}\right)^{\frac{1}{3}}}{\frac{1}{2} \left(1 - \frac{40}{95}\right)^{\frac{1}{6}} + \frac{1}{2} \left(1 - \frac{40}{125}\right)^{\frac{1}{3}}}$$

for  $t = 1, 2, 3, 4, 5, 6$  we get the values

$t$	${}_t p_{40}$
1	0.9965130
2	0.9929864
3	0.9894191
4	0.9858098
5	0.9821573
6	0.9784603

multiplying these by the radix gives the values of  $l_x$ , and taking the difference between successive values of  $l_x$  gives the values of  $d_x$ , so the life table is

$x$	$l_x$	$d_x$
40	10000.00	34.87
41	9965.13	35.27
42	9929.86	35.67
43	9894.19	36.09
44	9858.10	36.52
45	9821.57	36.97

[The discrepancy between  $l_{49}$ ,  $l_{50}$ , and  $d_{49}$  is caused by rounding.]

5. Using the lifetable:

$x$	$l_x$	$d_x$
50	10000.00	31.10
51	9968.90	34.75
52	9934.15	38.84
53	9895.30	43.44
54	9851.86	48.59
55	9803.28	54.35

calculate the probability that an individual aged 50 years and three months survives another 4 years, using:

(a) the uniform distribution of deaths assumption.

Using UDD, we get

$$l_{50\frac{1}{4}} = \frac{1}{4}l_{51} + \frac{3}{4}l_{50} = \frac{39968.90}{4}$$

and

$$l_{54\frac{1}{4}} = \frac{1}{4}l_{55} + \frac{3}{4}l_{54} = \frac{39358.86}{4}$$

The probability that the individual survives 4 years is therefore

$$\frac{l_{54\frac{1}{4}}}{l_{50\frac{1}{4}}} = \frac{39358.86}{39968.90} = 0.9847371$$

(b) the constant force of mortality assumption.

Using constant force of mortality we have between ages  $x$  and  $x + 1$ ,

$$l_{x+\alpha} = l_x e^{-\mu_x \alpha} = l_x (e^{-\mu_x})^\alpha = l_x \left( \frac{l_{x+1}}{l_x} \right)^\alpha = l_x^{1-\alpha} l_{x+1}^\alpha$$

This gives

$$l_{50\frac{1}{4}} = l_{51}^{\frac{1}{4}} l_{50}^{\frac{3}{4}} = 9992.216$$

and

$$l_{54\frac{1}{4}} = l_{55}^{\frac{1}{4}} l_{54}^{\frac{3}{4}} = 9839.692$$

The probability of surviving 4 years is therefore  $\frac{9839.692}{9992.216} = 0.9847358$ .

## Standard Questions

6. An insurance company wants to use a model of mortality of the form  $\mu_x = \frac{a}{100-x} + \frac{b}{125-x}$ . Based on the company's data, an individual aged 65 has probability 0.874 of surviving to age 80, and the probability of an individual aged 45 surviving to age 65 is 0.921. It is extremely important for these properties to match the observed data. The company chooses the values of  $a$  and  $b$  to ensure that these observations are matched by the model. What values of  $a$  and  $b$  should they choose?

We have that

$$\begin{aligned} \log(S_x(t)) &= - \int_x^{x+t} \mu_s ds = - \int_x^{x+t} \left( \frac{a}{100-s} + \frac{b}{125-s} \right) ds \\ &= a [\log(100-s)]_x^{x+t} + b [\log(125-s)]_x^{x+t} \\ &= a (\log(100-x-t) - \log(100-x)) + b (\log(125-x-t) - \log(125-x)) \end{aligned}$$

Substituting the given values

$$S_{45}(20) = 0.921$$

$$S_{65}(15) = 0.874$$

we get

$$a \log\left(\frac{35}{55}\right) + b \log\left(\frac{60}{80}\right) = \log(0.921)$$

$$a \log\left(\frac{20}{35}\right) + b \log\left(\frac{45}{60}\right) = \log(0.874)$$

$$b \left( \log\left(\frac{45}{60}\right) - \frac{\log\left(\frac{60}{80}\right) \log\left(\frac{20}{35}\right)}{\log\left(\frac{35}{55}\right)} \right) = \log(0.874) - \frac{\log(0.921) \log\left(\frac{20}{35}\right)}{\log\left(\frac{35}{55}\right)}$$

$$b = -0.4785434$$

$$a = 0.4866611$$

7. An insurance company prepares the following lifetable for an individual.

$x$	$l_x$	$d_x$
35	10000.00	23.49
36	9976.51	27.30
37	9949.21	31.76
38	9917.44	37.00
39	9880.44	43.14
40	9837.30	50.33

After an examination, it determines that the individual's probability of death in each year should be twice as much as that given in this table.

Prepare a new life table for this individual over the same range using radix 10,000.

We calculate  $q_x = \frac{d_x}{l_x}$  then double this value. For the new lifetable we use  $d_x = l_x q_x$ . For example, we get

$$d_{35} = \frac{2 \times 23.49}{10000} \times 10000 = 46.98$$

$$d_{36} = \frac{2 \times 27.30}{9976.51} \times 9953.02 = 54.57$$

$$d_{37} = \frac{2 \times 31.76}{9949.21} \times 9898.55 = 63.20$$

$$d_{38} = \frac{2 \times 37.00}{9917.44} \times 9835.35 = 73.39$$

$$d_{39} = \frac{2 \times 43.14}{9880.44} \times 9761.96 = 85.25$$

$$d_{40} = \frac{2 \times 50.33}{9837.30} \times 9676.71 = 99.02$$

The lifetable is therefore:

$x$	$l_x$	$d_x$
35	10000.00	46.98
36	9953.02	54.47
37	9898.55	63.20
38	9835.35	73.39
39	9761.96	85.25
40	9676.71	99.02