ACSC/STAT 3720, Life Contingencies I
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Toby Kenney
Homework Sheet 4
Model Solutions

Basic Questions

1. *Using the lifetable in Table 1, calculate* $\ddot{a}_{[38]+3}$ *at interest rate* $i = 0.04$. *You are given that* $A_{[38]+3} = 0.148816$.

   We have
$$\ddot{a}_{[38]+3} = \frac{1+i}{i}(1 - A_{[38]+3}) = 21 \times (1 - 0.148816) = 17.87486$$

2. *An individual aged 51 for whom Table 1 is appropriate, takes out a 5-year endowment insurance policy. The annual premiums are \$42,000, payable at the begining of each year. If the current interest rate is* $i = 0.06$, *what is the expected present value of the premiums paid?*

   We use our standard recurrence $A_{x:\overline{t+1}|} = (p_x A_{x+1:\overline{t}|} + q_x)(1+i)^{-1}$, starting from $A_{56:\overline{0}|} = 1$ to get

$$\begin{aligned}
A_{55:\overline{1}|} &= (1.06)^{-1} = 0.943396\\
A_{54:\overline{2}|} &= (1.06)^{-1}\left(0.943396 \times \frac{9823.08}{9838.38} + \frac{15.30}{9838.38}\right) = 0.890080\\
A_{53:\overline{3}|} &= 0.839845\\
A_{52:\overline{4}|} &= 0.792504\\
A_{51:\overline{5}|} &= 0.747880
\end{aligned}$$

   We then use our formula
$$\ddot{a}_{51:\overline{5}|} = \frac{1.06}{0.06} \times (1 - A_{51:\overline{5}|}) = \frac{1.06}{0.06}(1 - 0.747880) = 4.45412$$
   .

   The expected value of the premiums paid is therefore
$$4.45412 \times 42000 = \$187,073.04$$

   [An alternative approach is to calculate $\ddot{a}_{51:\overline{5}|}$ directly using the recurrence $\ddot{a}_{x:\overline{n+1}|} = 1 + p_x \ddot{a}_{x+1:\overline{n}|}(1+i)^{-1}$, starting from $\ddot{a}_{55:\overline{1}|} = 1$.

3. *An annuity pays out continuously at a rate of \$5,000 a year until the death of an individual currently aged 70 to whom the ultimate part of Table 1 applies. What is the expected present value of this annuity, using the uniform distribution of deaths assumption, and force of interest* $\delta = 0.04$? *You are given the following values of* A_{70} *at various interest rates:*

i	A_{70}
.03922071	0.40409
0.04	0.39769
0.040811	0.391169
0.04879016	0.333917
0.05	0.326225

$\delta = 0.04$ corresponds to $i = e^{0.04} - 1 = 0.040811$. From the above table, this gives $A_{70} = 0.391169$. Using the uniform distribution of deaths assumption, we have

$$\overline{A}_{70} = \frac{i}{\delta} A_{70} = \frac{0.04081077}{0.04} \times 0.391169 = 0.3990977$$

We then use

$$\bar{a}_{70} = \frac{1 - \overline{A}_{70}}{\delta} = \frac{1 - 0.3990977}{0.04} = 15.02256$$

The EPV of the annuity payments is therefore

$$15.02256 \times 5000 = \$75,112.79$$

4. *A pension plan pays monthly benefits of \$6,000 to an individual aged 65. What is the expected present value of the benefit under the uniform distribution of deaths assumption, interest rate* $i^{(12)} = 0.03$ *and the lifetable in Table 1? [These allow us to calculate* $A_{65} = 0.431098$.*]*

   Under the UDD assumption, we have

$$A_{65}^{(12)} = \frac{i}{i^{(12)}} A_{65} = \frac{0.03041596}{0.03} \times 0.431098 = 0.4370753$$

   We have $d^{(12)} = 12\left(1 - \left(1 + \frac{i^{(12)}}{12}\right)^{-1}\right) = 0.02992519$, so we can use

$$\ddot{a}_{65}^{(12)} = \frac{1 - A_{65}^{(12)}}{d^{(12)}} = \frac{1 - 0.4370753}{0.02992519} = 18.81107$$

   The EPV of pension benefits is therefore

$$6000 \times 12 \times 18.81107 = \$1,337,597.04$$

Standard Questions

5. *A pension plan pays an annual benefit of \$24,000 to an individual aged 61, for whom the ultimate part of the lifetable in Table 1 applies. The interest rate is* $i = 0.05$*, which gives* $A_{61} = 0.231363$ *and* $A_{71} = 0.338212$*. Payments are guaranteed for the first 10 years. The individual wants to remove this guarantee (so that benefits will stop immediately upon death), and keep the EPV of the benefits the same. What should the new annual payments be?*

   The EPV of the pension with guaranteed benefits is

$$24000\left(\ddot{a}_{\overline{10}|0.05} + {}_{10}p_{61}(1.05)^{-10}\ddot{a}_{71}\right) = 24000\left(\frac{1.05}{0.05}(1 - 1.05^{-10}) + \frac{9269.88}{9697.28} \times (1.05)^{-10} \times \frac{1.05}{0.05} \times (1 - 0.338212)\right)$$

Once the guaranteed benefit is removed, we have

$$\ddot{a}_{71} = \frac{1.05}{0.05}(1 - 0.231363) = 16.14138$$

, so to get the same EPV, the annual payments should be

$$\frac{390328.18}{16.14138} = \$24,181.84$$

6. *A man aged 104, to whom the ultimate part of the lifetable in Table 1 applies, wants a pension which will pay \$50,000 in a year's time, and thereafter will provide annual payments increasing by 4% every year (so the second payment when the man turns 106 will be \$52,000). What is the expected present value of the benefits of this pension if the current interest rate is $i = 0.03$?*

   Payments are increasing by 4% per year, so the "real" rate of interest is given by $\frac{0.03-0.04}{1.04} = -0.009615385$. We use this rate of interest and the standard recurrence starting from $A_{125} = 1$ to get

$$\begin{aligned}
A_{125} &= 1 \\
A_{124} &= 1.00971 \\
A_{123} &= 1.0108 \\
A_{122} &= 1.01236 \\
A_{121} &= 1.01353 \\
A_{120} &= 1.01472 \\
A_{119} &= 1.01599 \\
A_{118} &= 1.01734 \\
A_{117} &= 1.0188 \\
A_{116} &= 1.02036 \\
A_{115} &= 1.02203 \\
A_{114} &= 1.02383 \\
A_{113} &= 1.02575 \\
A_{112} &= 1.0278 \\
A_{111} &= 1.02999 \\
A_{110} &= 1.03232 \\
A_{109} &= 1.0348 \\
A_{108} &= 1.03744 \\
A_{107} &= 1.04024 \\
A_{106} &= 1.04321 \\
A_{105} &= 1.04635
\end{aligned}$$

   From this we calculate

$$\ddot{a}_{105} = \frac{1+i}{i}(1 - A_{105}) = \frac{1 - 0.009615385}{-0.009615385} \times (1 - 1.04635) = 4.77405$$

So in one year's time, the EPV of the pension will be $4.77405 \times 50000 = 238702.5$. The current EPV is therefore $238702.5 p_{104}(1.03)^{-1} = 238702.5 \times \frac{1755.27}{2055.64}(1.03)^{-1} = \$197,886.70$.

7. *A woman aged 62 is receiving a monthly pension of \$20,000 at the start of each month. She wants to change this to an annual pension. If the appropriate life table is the ultimate part of Table 1 and the interest rate is $i = 0.06$, then we can calculate $A_{62} = 0.190481$. Use Woolhouse's formula to calculate the annual pension that has the same expected present value. [You may use the approximation $\mu_x = \frac{1}{2}(q_x + q_{x-1})$.]*

   We have $\ddot{a}_{62} = \frac{1.06}{0.06}(1 - 0.190481) = 14.3015$. Now using the given approximation, we get $\mu_{62} \approx \frac{1}{2}(q_{62} + q_{61}) = 1 - \frac{1}{2}\left(\frac{9638.51}{9669.17} + \frac{9669.17}{9697.28}\right) = 0.003034827$. We also have $\delta = \log(1.06) = 0.05826891$. Woolhouse's formula then gives

$$\ddot{a}_{62}^{(12)} = \ddot{a}_{62} - \frac{11}{24} - \frac{143}{1728}(0.05826891 + 0.003034827) = 13.83809$$

   The annual pension with the same present value as a monthly pension of \$20,000 is therefore

$$\frac{20000 \times 12 \times 13.83809}{14.3015} = \$232,223.31$$

Table 1: Select lifetable to be used for questions on this assignment

x	$l_{[x]}$	$l_{[x]+1}$	$l_{[x]+2}$	$l_{[x]+3}$
25	9998.75	9997.65	9996.30	9994.66
26	9997.00	9995.83	9994.40	9992.66
27	9995.14	9993.90	9992.38	9990.52
28	9993.16	9991.84	9990.22	9988.24
29	9991.05	9989.65	9987.92	9985.80
30	9988.81	9987.30	9985.46	9983.18
31	9986.40	9984.80	9982.82	9980.38
32	9983.83	9982.11	9979.99	9977.37
33	9981.07	9979.23	9976.95	9974.13
34	9978.11	9976.13	9973.68	9970.64
35	9974.93	9972.79	9970.16	9966.88
36	9971.50	9969.20	9966.36	9962.82
37	9967.80	9965.33	9962.25	9958.44
38	9963.81	9961.14	9957.82	9953.69
39	9959.50	9956.61	9953.02	9948.55
40	9954.84	9951.71	9947.82	9942.98
41	9949.79	9946.41	9942.19	9936.94
42	9944.32	9940.66	9936.08	9930.38
43	9938.39	9934.41	9929.45	9923.26
44	9931.96	9927.64	9922.25	9915.52
45	9924.97	9920.28	9914.42	9907.10
46	9917.37	9912.28	9905.91	9897.94
47	9909.11	9903.58	9896.65	9887.98
48	9900.13	9894.11	9886.57	9877.13
49	9890.36	9883.80	9875.59	9865.30
50	9879.71	9872.57	9863.63	9852.42
51	9868.12	9860.34	9850.59	9838.38
52	9855.48	9847.01	9836.39	9823.08
53	9841.72	9832.48	9820.90	9806.39
54	9826.71	9816.64	9804.02	9788.18
55	9810.34	9799.37	9785.60	9768.33
56	9792.49	9780.52	9765.51	9746.67
57	9773.03	9759.97	9743.60	9723.05
58	9751.79	9737.56	9719.69	9697.28
59	9728.63	9713.10	9693.62	9669.17
60	9703.36	9686.43	9665.17	9638.51
61	9675.80	9657.33	9634.15	9605.07
62	9645.73	9625.59	9600.31	9568.61
63	9612.94	9590.98	9563.42	9528.85
64	9577.18	9553.24	9523.19	9485.52
65	9538.19	9512.09	9479.35	9438.30
66	9495.69	9467.25	9431.58	9386.86
67	9449.37	9418.39	9379.54	9330.85
68	9398.90	9365.17	9322.87	9269.88
69	9343.95	9307.23	9261.20	9203.55
70	9284.12	9244.18	9194.11	9131.43
71	9219.03	9175.59	9121.17	9053.07
72	9148.24	9101.03	9041.91	8967.97
73	9071.30	9020.03	8955.85	8875.63

x	$l_{[x]}$	$l_{[x]+1}$	$l_{[x]+2}$	$l_{[x]+3}$
74	8987.73	8932.10	8862.49	8775.52
75	8897.04	8836.71	8761.27	8667.10
76	8798.69	8733.34	8651.66	8549.78
77	8692.13	8621.41	8533.09	8423.00
78	8576.81	8500.36	8404.95	8286.16
79	8452.13	8369.60	8266.68	8138.66
80	8317.52	8228.53	8117.67	7979.93
81	8172.36	8076.57	7957.35	7809.41
82	8016.08	7913.13	7785.15	7626.56
83	7848.11	7737.67	7600.54	7430.89
84	7667.89	7549.66	7403.05	7221.99
85	7474.92	7348.64	7192.27	6999.51
86	7268.77	7134.21	6967.86	6763.22
87	7049.07	6906.07	6729.62	6513.04
88	6815.55	6664.05	6477.46	6249.02
89	6568.09	6408.10	6211.48	5971.42
90	6306.70	6138.35	5931.96	5680.73
91	6031.59	5855.15	5639.41	5377.67
92	5743.19	5559.08	5334.61	5063.27
93	5442.15	5250.97	5018.61	4738.86
94	5129.44	4931.97	4692.79	4406.12
95	4806.33	4603.54	4358.89	4067.08
96	4474.39	4267.51	4018.96	3724.10
97	4135.60	3926.04	3675.44	3379.91
98	3792.25	3581.66	3331.11	3037.57
99	3447.02	3237.23	2989.05	2700.39
100	3102.90	2895.94	2652.63	2371.88
101	2763.19	2561.21	2325.37	2055.64
102	2431.39	2236.61	2010.90	1755.27
103	2111.15	1925.80	1712.81	1474.18
104	1806.12	1632.34	1434.48	1215.44
105	1519.82	1359.55	1178.94	981.65
106	1255.46	1110.36	948.70	774.71
107	1015.81	887.14	745.58	595.71
108	802.96	691.49	570.56	444.87
109	618.23	524.17	423.71	321.41
110	462.04	385.00	304.13	223.65
111	333.80	272.80	210.00	149.10
112	231.99	185.53	138.71	94.62
113	154.19	120.34	87.07	56.74
114	97.30	73.90	51.50	31.84
115	57.78	42.55	28.41	16.52
116	31.92	22.69	14.43	7.81
117	16.15	11.04	6.63	3.30
118	7.34	4.79	2.69	1.21
119	2.90	1.79	0.93	0.37
120	0.95	0.55	0.26	0.09
121	0.23	0.13	0.05	0.01
122	0.03	0.02	0.01	0.00