MATH/STAT 3720, Life Contingencies I
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In Class Examples

Introduction to Life Insurance

Term Life Insurance Contract

- Policyholder pays regular premiums for the term of the policy, or until death.
- If policyholder dies, insurance pays death benefit.

Whole Life Insurance Contract

- Policyholder pays regular premiums until death, or an agreed age (usually 80).
- When policyholder dies, insurance pays death benefit.

Endowment Insurance

- Policyholder pays regular premiums until death, or an agreed future time.
- At death, or the agreed future time, insurance pays agreed benefit.

Annuities

- Policyholder pays a lump sum
- Insurance pays regular payments until policyholder dies.

Participating vs. Non-participating

Participating Life Insurance

Amount by which premiums and investment gains exceed liabilities and costs is returned to policyholders in one of a number of ways:

- Cash dividends
- Reduced premiums
- Increased policy value

Modern Insurance Contracts

- Universal Life Insurance
- Unitized with-profit
- Equity-linked Insurance

Underwriting

The process of acquiring information about factors which affect the potential policyholder's risk, and determining an appropriate premium. Relevant information might include:

- Age
- Sex
- Medical history (personal & family)
- Smoking habits
- Occupation
- Dangerous hobbies

For large sums insured, it may also involve a medical examination.

Policyholders will generally be classified into the following categories:

Preferred lives	very low mortality risk
Normal lives	may have some higher risk factors
Rated lives	higher than average risk, can be insured for higher premium
Uninsurable lives	very high risk, insurer will not enter insurance contract at any price.

Annuities

- Single Premium Deferred Annuity
- Single Premium Immediate Annuity
- Regular Premium Deferred Annuity
- Joint Life Annuity
- Last Survivor Annuity
- Reversionary Annuity

Other Insurance Contracts

- Income Protection insurance
- Critical Illness insurance
- Long-term care insurance
- Hospital Indemnity insurance

Pensions

Defined Benefit

Annuity upon retirement defined by a formula. For example,

$$\text{Final Salary} \times \text{Years of Service} \times \text{Accrual Rate}$$

Paid for by regular contributions by employer and (usually) employee.

Defined Contribution

Employer and employee pay pre-determined contribution into fund. Upon retirement, value of fund is available to employee to provide retirement income.

Actuarial Tasks

- Calculate premiums, taking into account:
  - Probability of death/disability or other claim event
  - Premiums and benefits
  - Investment gains
- Calculate dividends
- Calculate surrender value of policy
- Ensure adequate reserves to cover risk
- Invest money to match future liabilities

Question 1

The probability that a newborn baby will live to age 34 is 0.983. The probability that the same newborn baby will live to age 46 is 0.964. What is the probability that a life aged 34, with the same characteristics, will survive for a further 12 years?

Properties of Survival Functions

Properties of Valid Survival Functions

- $S_x(0) = 1$
- $S_x(t)$ is a decreasing function.
- $\lim_{t\to\infty} S_x(t) = 0$

Additional Assumptions for Future Lifetime

- $S_x(t)$ is differentiable for all $t > 0$.
- $\lim_{t\to\infty} tS_x(t) = 0$.
- $\lim_{t\to\infty} t^2 S_x(t) = 0$.

Force of Mortality

Definition

For a random life, the *force of mortality* at age x is given by

$$\mu_x = \frac{d}{dt}(S_x(t))|_{t=0}$$

Other Formulae

- $\mu_x = \frac{f_0(x)}{S_0(x)}$
- $S_0(x) = e^{-\int_0^x \mu_x dx}$
- $S_x(t) = e^{-\int_x^{x+t} \mu_s ds}$

Question 2

Suppose lifetime is modelled as

$$F_0(t) = 1 - \left(1 - \frac{x}{130}\right)^{\frac{1}{4}}$$

Calculate the force of mortality.

Question 3

The Gompertz law of mortality states that $\mu_x = Bc^x$ for two constants B and c, where $c > 1$. Calculate the survival function $S_0(x)$ based on this law.

2.4 Actuarial Notation

Notation

- Survival probability ${}_tp_x = S_x(t)$
- Mortality probability ${}_tq_x = F_x(t)$
- Deferred mortality probability ${}_{u|t}q_x = S_x(u) - S_x(t+u)$

Relations

$$
\begin{aligned}
{}_tp_x + {}_tq_x &= 1 \\
{}_{u|t}q_x &= {}_up_x - {}_{u+t}p_x \\
{}_{u+t}p_x &= {}_up_x \, {}_tp_{x+u}
\end{aligned}
$$

$$
\begin{aligned}
\mu_x &= -\frac{1}{{}_xp_0}\frac{d}{dx}({}_xp_0) \\
f_x(t) &= {}_tp_x \mu_{x+t} \\
{}_tq_x &= \int_0^t {}_sp_x \mu_{x+s} ds
\end{aligned}
$$

Question 4

Suppose mortality follows a Gompertz law with $B = 0.005$ and $c = 1.07$. Calculate the following:

(a) p_{36}

(b) ${}_3q_{72}$

(c) ${}_{2|2}q_{29}$

Question 5

Suppose mortality follows a Gompertz law with $B = 0.007$ and $c = 1.06$.
(a) Calculate the exact value of q_{57}, and compare it with $\mu_{57.5}$.
(b) Calculate the exact value of q_{87} and compare it with $\mu_{87.5}$.

Question 6

Suppose lifetime is modelled as

$$F_0(t) = 1 - \left(1 - \frac{x}{130}\right)^{\frac{1}{4}}$$

(a) Calculate $\mathring{e}_{39}$.
(b) Calculate $\mathrm{Var}(T_{39})$.

Question 7

Suppose lifetime is modelled as

$$F_0(t) = 1 - \left(1 - \frac{x}{130}\right)^{\frac{1}{4}}$$

Calculate $\mathring{e}_{39:\overline{25}|}$

Question 8

Future lifetime for a particular life is modelled as following a Gompertz law with $B = 0.0001$ and $c = 1.06$. The life is currently aged 47. What is the probability that the individual's curtate future lifetime is 6?

Question 9

Future lifetime for a particular life is modelled as following a Gompertz law with $B = 0.0001$ and $c = 1.06$. The life is currently aged 47. What is the individual's curtate expected lifetime?

Question 10

Suppose lifetime is modelled as

$$F_0(t) = 1 - \left(1 - \frac{x}{130}\right)^{\frac{1}{4}}$$

Calculate $\mathring{e}_x - (e_x + 0.5)$.

Format of life tables

A lifetable is a compact representation of the annual survival probabilities of a mortality model in a certain age range.

- The first column l_x gives the number of lives still alive a the given age.
- The second column d_x gives the number of lives who die at a given age.
- The number of lives alive at the starting age is called the radix.
- Probabilities for a life aged x are calculated by dividing by l_x.

Question 11

The following is an extract from a life table:

x	l_x	d_x
40	10000.00	47.12
41	9952.88	49.46
42	9903.42	50.93
43	9852.49	52.40
44	9800.09	55.88
45	9744.21	59.94
46	9684.27	63.73
47	9621.54	67.01
48	9554.53	70.66
49	9483.87	72.78
50	9411.09	74.40

Calculate:

(a) ${}_6p_{42}$

(b) q_{46}

(c) ${}_{4|2}q_{41}$

(d) $e_{43:\overline{5}|}$

Question 12

Compute a lifetable starting with age 30 and radix 10,000, ending at age 40, using a Makeham model of mortality $\mu_x = A + Bc^x$ with $A = 0.0002$, $B = 0.0002$ and $c = 1.06$. [You may use $\mu_{x+0.5}$ as an approximation for q_x.]

Solution to Question 12

x	l_x	d_x
30	10000.00	13.83
31	9986.17	14.52
32	9971.66	15.25
33	9956.41	16.02
34	9940.40	16.83
35	9923.57	17.69
36	9905.88	18.60
37	9887.28	19.56
38	9867.72	20.57
39	9847.14	21.64
40	9825.50	22.78

Uniform Distribution of Deaths

- In this model we assume that conditional on dying aged x, the exact age at death is uniformly distributed between x and $x+1$.
- That is ${}_tq_x = tq_x$ for $0 < t < 1$.
- Recall that

$$\mu_{x+t} = \frac{d}{dt}\left(\frac{{}_tq_x}{1 -_t q_x}\right) = \frac{q_x}{1 - tq_x} + \frac{t{q_x}^2}{(1 - tq_x)^2}$$

so under this model force of mortality is increasing throughout the year, though this increase is usually less than the true increase.

Question 13

Under a certain model of mortality, we have $q_{36} = 0.0004$. Using the uniform distribution of deaths assumption, what is ${}_{0.6}q_{36.3}$?

Question 14

The following is an extract from a life table:

x	l_x	d_x
40	10000.00	47.12
41	9952.88	49.46
42	9903.42	50.93
43	9852.49	52.40
44	9800.09	55.88
45	9744.21	59.94
46	9684.27	63.73
47	9621.54	67.01
48	9554.53	70.66
49	9483.87	72.78
50	9411.09	74.40

An individual aged 42 and 4 months wishes to purchase a term life insurance contract for 6 years. What is the probability that the individual dies during this contract?

3.3 Fractional Age Assumptions

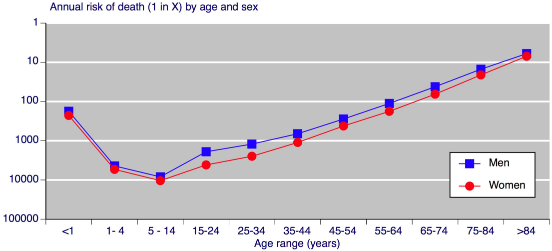
Constant Rate of Mortality

- In this model we assume that force of mortality is constant between x and $x+1$.
- This gives ${}_tp_x = e^{-\mu t}$ for $0 < t < 1$, where μ is chosen to give the correct value of p_x. That is $\mu = -\log(p_x)$.

Question 15

Under a certain model of mortality, we have $q_{36} = 0.0004$. Using the constant rate of mortality assumption, what is ${}_{0.6}q_{36.3}$?

Figure: Mortality Rates by Age from UK Mortality statistics 2005



Source: http://www.medicine.ox.ac.uk/bandolier/booth/Risk/dyingage.html

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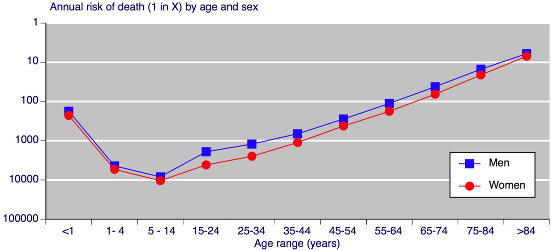

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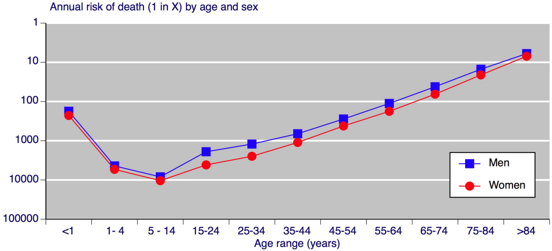
- High mortality just after birth.

Figure: Mortality Rates by Age from UK Mortality statistics 2005



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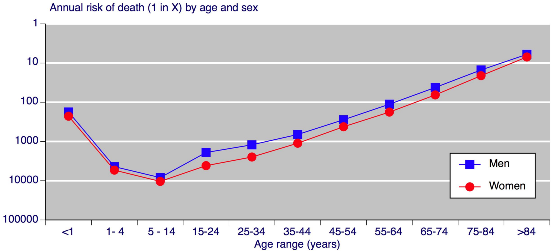
- High mortality just after birth.
- Mortality rate drops quickly, continues decreasing till age 10.

Figure: Mortality Rates by Age from UK Mortality statistics 2005



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- High mortality just after birth.
- Mortality rate drops quickly, continues decreasing till age 10.
- Male and female mortalities diverge significantly in late teens

3.4 National Life Tables

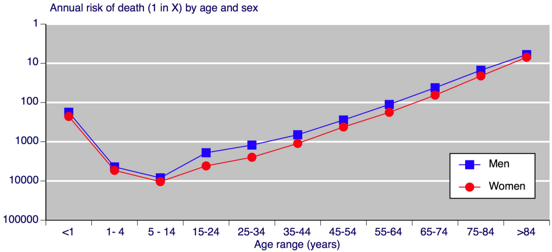

Figure: Mortality Rates by Age from UK Mortality statistics 2005

Source: http://www.medicine.ox.ac.uk/bandolier/booth/Risk/dyingage.html

- High mortality just after birth.
- Mortality rate drops quickly, continues decreasing till age 10.
- Male and female mortalities diverge significantly in late teens
- Mortality increases after age 10, accident hump in late teens.

3.4 National Life Tables

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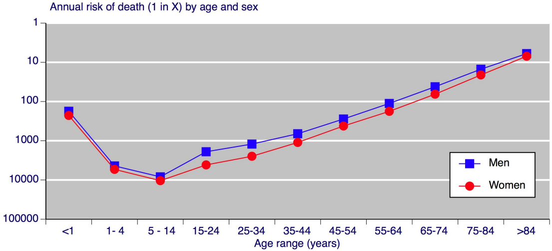

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- High mortality just after birth.
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- Male and female mortalities diverge significantly in late teens
- Mortality increases after age 10, accident hump in late teens.
- Rates for females are lower than for males.

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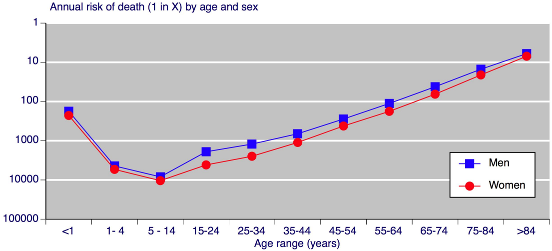

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- High mortality just after birth.
- Mortality rate drops quickly, continues decreasing till age 10.
- Male and female mortalities diverge significantly in late teens
- Mortality increases after age 10, accident hump in late teens.
- Rates for females are lower than for males.
- Gompertz model fits well for large ages, badly for young ages.

Question 16

Why are mortality rates lower for policyholders than members of the general population of the same age and sex? Should you purchase a life insurance policy to reduce your mortality?

Answer to Question 16

- People who buy life insurance are better off financially.
- People who buy life insurance are usually married, often with families.
- In order to buy life insurance individuals must be in good health.

3.7 Select and Ultimate Survival Models

Model

- Idea is to incorporate the impact of underwriting into the mortality model.
- Ultimate model μ_x represents the usual mortality for a policyholder with given characteristics.
- Selection period represents the time since underwriting for which mortality is better than normal.
- During this period, an individual has select mortality given by $\mu_{[x]+t} = D^{s-t}\mu_{x+t}$ where x is the age at which the life was select (i.e. the age at the time of the underwriting), $x + t$ is the current age and s is the selection period.
- While we usually have $D < 1$ to represent lower mortality in the selection period, the model could in principle be used with $D > 1$ to represent short-term health risks.

Question 17

Using a model where ultimate mortality follows Makeham's model $\mu_x = 0.00022 + 0.0000027 \times 1.124^x$, and the select mortality for an individual selected s years ago is given by $\mu_{[x]+s} = 0.9^{2-s}\mu_{x+s}$. Calculate the probability that an individual currently aged 42, and select at age 41 will live to age 48. [You may use the approximation $q_{[x]+t} \approx \mu_{[x+0.5]+t}$.]

3.9 Select Life Tables

Format of a Select Life Table

- Usually have columns $l_{[x]}, l_{[x]+1}, \ldots, l_{[x]+s}$ where s is the selection period.
- No column d_x giving mortality.
- Final column gives the ultimate mortality model.
- If table starts at age x, radix is $l_x = l_{[x-s]+s}$, in the final column. Note that this value does not appear in the table.

Question 18

Use the standard select survival model from Question 17 to construct a select life table with $l_{28} = 100000$, for ages between 30 and 40 at time of selection.

Answer to Question 18

x	$l_{[x]}$	$l_{[x]+1}$	$l_{[x]+2}$
30	99991.33	99967.40	99939.96
31	99961.50	99936.81	99908.44
32	99930.70	99905.17	99875.74
33	99898.82	99872.34	99841.73
34	99865.72	99838.17	99806.23
35	99831.26	99802.51	99769.07
36	99795.26	99765.16	99730.03
37	99757.53	99725.92	99688.90
38	99717.86	99684.55	99645.40
39	99676.01	99640.79	99599.25
40	99631.71	99594.34	99550.12

Question 19

Using the following select life table

x	$l_{[x]}$	$l_{[x]+1}$	$l_{[x]+2}$	$l_{[x]+3}$
25	99964.85	99938.56	99907.27	99869.96
26	99921.59	99894.99	99863.28	99825.45
27	99877.76	99850.82	99818.66	99780.23
28	99833.32	99805.99	99773.33	99734.25
29	99788.19	99760.44	99727.22	99687.41
30	99742.30	99714.07	99680.24	99639.61
31	99695.56	99666.81	99632.28	99590.76
32	99647.88	99618.54	99583.25	99540.74
33	99599.16	99569.17	99533.03	99489.41

Calculate the probability that an individual aged 27, who has been select for 1 year, will die at age 31 or 32.

4.4 Valuation of Life Insurance Benefits

Notation

i	Effective annual interest rate
v	Annual discount factor $(1+i)^{-1}$
δ	Force of interest $\log(1+i)$
$i^{(p)}$	Nominal interest rate compounded p times per year
d	Annual discount rate $1-v$
$d^{(p)}$	Nominal discount rate compounded p times per year $p(1-v^{\frac{1}{p}})$
$\bar{A}_x$	Expected present value of \$1 when a life of present age x dies
A_x	Expected present value of \$1 at the end of the year in which a life of present age x dies
$A_x^{(m)}$	Expected present value of \$1 at the end of the period $\frac{1}{m}$th of a year in which a life of present age x dies

Question 20

Suppose the future lifetime random variable satisfies

$$F_0(x) = 1 - \left(1 - \frac{x}{130}\right)^{\frac{1}{4}}$$

and force of interest is given by $\delta = 0.04$.
(a) Calculate $\overline{A}_{37}$.
(b) Calculate the variance of the present value of a payment of $1 immediately at the time the life dies.

Question 21

Suppose the future lifetime random variable satisfies

$$F_0(t) = 1 - \left(1 - \frac{x}{130}\right)^{\frac{1}{4}}$$

and force of interest is given by $\delta = 0.05$.
(a) Calculate A_{43}.
(b) Calculate $A_{43}^{(12)}$.

Variance of Present Value of Future Payments

- Expected present value of \$1 at time of death is

$$\int_0^\infty {}_tp_x \mu_x e^{-\delta t}\,dt$$

- Expected square of present value of \$1 at time of death is

$$\int_0^\infty {}_tp_x \mu_x \left(e^{-\delta t}\right)^2 dt = \int_0^\infty {}_tp_x \mu_x e^{-2\delta t}\,dt$$

- This is just the EPV evaluated at force of interest 2δ. We denote this ${}^2\overline{A}_x$.
- The same applies to annual annuities.

Question 22

The following is an excerpt from a life table

x	l_x	d_x
30	10000.00	3.49
31	9996.51	3.60
32	9992.90	3.72
33	9989.18	3.86
34	9985.32	4.00
35	9981.32	4.16
36	9977.16	4.33
37	9972.82	4.52
38	9968.30	4.73
39	9963.57	4.96

You are given that $A_{40} = 0.1211$, ${}^2A_{40} = 0.03071$, and $\delta = 0.06$. Calculate A_{32} and the variance of a payment of \$1 when a life currently aged 32 dies.

Recurrence relation for A_x

- By definition we have $A_x = q_x(1+i)^{-1} + p_x q_{x+1}(1+i)^{-2} + \cdots$.
- Recall that ${}_{n-1}p_x = p_x p_{x+1} p_{x+2} \cdots p_{x+n-2}$. So we can factorise $A_x = (1+i)^{-1}\left(q_x + p_x\left(q_{x+1}(1+i)^{-1} + p_{x+1}q_{x+2}(1+i)^{-2} + \cdots\right)\right)$
- Note that the term in the brackets is exactly A_{x+1}, so we can write this as a recurrence $A_x = (1+i)^{-1}(q_x + p_x A_{x+1})$.
- We get the same recurrences for term and endowment insurances. Note that the term decreases by one year each time:

$$A^1_{x:\overline{n}|} = (1+i)^{-1}\left(q_x + p_x A^1_{x+1:\overline{n-1}|}\right)$$

$$A_{x:\overline{n}|} = (1+i)^{-1}\left(q_x + p_x A_{x+1:\overline{n-1}|}\right)$$

- Difference is starting values $A^1_{x:\overline{0}|} = 0$ and $A_{x:\overline{0}|} = 1$. For A_x, choose a starting value large enough that $p_x \approx 0$. For this recurrence the starting value is not very important.

Question 23

The following is an excerpt from a life table.

x	l_x	d_x
32	10000.00	4.84
33	9995.16	4.86
34	9990.30	4.89
35	9985.41	4.92
36	9980.49	4.95
37	9975.54	4.98
38	9970.55	5.02
39	9965.53	5.06
40	9960.48	5.10

Calculate $A^{1}_{33:\overline{5}|}$ at interest rate $v = 0.96$.

Question 24

The following is an excerpt from a life table.

x	l_x	d_x
30	10000.00	9.82
31	9990.18	10.34
32	9979.84	10.95
33	9968.89	11.64
34	9957.25	12.44
35	9944.80	13.37
36	9931.44	14.42
37	9917.01	15.64
38	9901.37	17.05
39	9884.32	18.66
40	9865.67	20.51

Calculate $A_{30:\overline{10}|}$ at interest rate $i = 0.07$.

Question 25

The following is an excerpt from a life table.

x	l_x	d_x
30	10000.00	9.82
31	9990.18	10.34
32	9979.84	10.95
33	9968.89	11.64
34	9957.25	12.44
35	9944.80	13.37
36	9931.44	14.42
37	9917.01	15.64
38	9901.37	17.05
39	9884.32	18.66
40	9865.67	20.51

Calculate $_{4|}A_{30:\overline{6}|}$ at interest rate $i = 0.07$.

4.5 Relating Different Cases of Whole Life Insurance

Uniform Distribution of Deaths

- The difference between A_x and $\overline{A}_x$ is that in the former, the payment is delayed to the end of the year. The effect of this delay is to multiply the present value by $e^{-\delta t}$, where t is the time by which payment is delayed.
- Under UDD, the amount of delay is uniformly distributed between 0 and 1. So if the present value of payment is B for the annual benefit, then the expected present value is $\int_0^1 e^{\delta t} B\, dt$ for the continuous benefit.
- Since the same factor applies to every year of death, we get

$$\overline{A}_x = \int_0^1 e^{\delta t}\, dt A_x = \left[\frac{e^{\delta t}}{\delta}\right]_0^1 A_x = \frac{e^{\delta t} - 1}{\delta} A_x = \frac{i}{\delta} A_x$$

- The same approach for $\frac{1}{m}$thly benefits gives $A_x^{(m)} = \frac{i}{i^{(m)}} A_x$.

Question 26

The following is an excerpt from a lifetable:

x	l_x	d_x
40	10000.00	6.63
41	9993.37	6.85
42	9986.51	7.09
43	9979.42	7.36
44	9972.06	7.65
45	9964.41	7.96
46	9956.45	8.31
47	9948.14	8.69
48	9939.45	9.11
49	9930.34	9.56
50	9920.78	10.06

Force of interest is $\delta = 0.035$. Calculate $\overline{A}^{1}_{40:\overline{10}|}$ using a uniform distribution of deaths assumption.

Question 27

The following is an excerpt from a life table:

x	l_x	d_x
35	10000.00	54.89
36	9945.11	63.26
37	9881.85	73.00
38	9808.85	84.30
39	9724.54	97.38
40	9627.16	112.48

Calculate $A^{(12)}{}^{\,1}_{35:\overline{5}|}$ using:

(a) Force of interest $\delta = 0.04$ and the Uniform Distribution of Deaths.
(b) Force of interest $\delta = 0.04$ and the claim acceleration method.
(c) Force of interest $\delta = 0.12$ and the Uniform Distribution of Deaths.
(d) Force of interest $\delta = 0.12$ and the claim acceleration method.

Question 28

An insurance policy pays a death benefit equal to the number of years the policy has been in force plus one. Using the life table below, calculate the EPV of a policy with a term of 5 years for someone who has just purchased the policy at age 43. The annual effective interest rate is $i = 6\%$.

x	l_x	d_x
41	10000.00	25.66
42	9974.34	25.61
43	9948.74	25.56
44	9923.17	25.52
45	9897.65	25.48
46	9872.17	25.44
47	9846.74	25.40
48	9821.34	25.36

Question 29

For the same insurance policy and lifetable as in Question 28, calculate the expected present value of a policy with an initial term of 7 years for an individual who purchased the policy two years ago at age 41. The annual effective interest rate is $i = 6\%$.

x	l_x	d_x
41	10000.00	25.66
42	9974.34	25.61
43	9948.74	25.56
44	9923.17	25.52
45	9897.65	25.48
46	9872.17	25.44
47	9846.74	25.40
48	9821.34	25.36

Geometrically Increasing Death Benefits

- If death benefit is $B(1+j)^t$, then EPV is $\sum_{n=1}^{\infty} {}_{n-1}p_x q_{x+n-1} B(1+j)^n (1+i)^{-n}$
- This is the same as the death benefit for a policy at the "real" rate of interest $i^* = \frac{1+i}{1+j} - 1$

Question 30

A woman aged 30 buys a house with a mortgage of $200,000. She amortises this amount with annual payments over a period of 8 years at $i = 6\%$. She takes out mortgage insurance, which pays off the outstanding balance (principle plus interest) of the mortgage at the end of the year in which she dies. [Assume that the mortgage company does not charge a penalty for early repayment in this case.] If the insurance company uses an interest rate $i = 5.6\%$ and the life table below, calculate the EPV of the benefit on this policy.

x	l_x	d_x
30	10000.00	7.25
31	9992.75	7.33
32	9985.42	7.41
33	9978.00	7.50
34	9970.50	7.60

x	l_x	d_x
35	9962.89	7.71
36	9955.18	7.83
37	9947.35	7.96
38	9939.40	8.10

Revision

- Present value of an annuity with n payments at interest rate i is $a_{\overline{n}|i} = \frac{1-(1+i)^{-n}}{i}$.
- For an annuity-due, we have $\ddot{a}_{\overline{n}|i} = \frac{1-(1+i)^{-n}}{d}$, where $d = 1 - (1+i)^{-1}$ is the discount rate.
- Present value of a perpetuity is $\frac{1}{i}$ or $\frac{1}{d}$ for a perpetuity-due.
- Helpful way to think of annuity due is a perpetuity whose payments stop after n years. That is, as the difference between two perpetuities, one beginning immediately, the other beginning after n years.

Formulae for Present Value of a Whole-Life Annuity-due

$$\ddot{a}_x = \frac{1 - A_x}{d}$$

$$\ddot{a}_x = \sum_{k=0}^{\infty} v^k \, {}_k p_x$$

$$\ddot{a}_x = \sum_{k=0}^{\infty} \ddot{a}_{\overline{k+1}|} \, {}_{k|}q_x$$

Variance of present value is given by:

$$\mathrm{Var}(Y) = \frac{{}^2A_x - (A_x)^2}{d^2}$$

Question 31

Suppose lifetime is modelled as

$$F_0(t) = 1 - \left(1 - \frac{x}{130}\right)^{\frac{1}{4}}$$

Using each of the formulae on the previous slide, calculate the EPV of an annual annuity due on a life currently aged 42, if the annual force of interest is given by $\delta = 0.04$.

Question 32

A certain insurance company uses the following life table:

x	l_x	d_x
60	10000.00	24.69
61	9975.31	26.76
62	9948.55	29.02
63	9919.53	31.49
64	9888.04	34.17
65	9853.87	37.10

x	l_x	d_x
66	9816.78	40.28
67	9776.50	43.74
68	9732.76	47.50
69	9685.26	51.58
70	9633.68	56.01

From their policy pricing, you determine that they evaluate $\ddot{a}_{60} = 19.64$ and $\ddot{a}_{70} = 11.20$. What rate of interest are they using for these calculations?

(a) 0.687% (b) 1.223% (c) 1.891% (d) 2.433%

Question 33

A certain insurance policy involves a term annuity with an annual payment in advance of $1,200 for a term of 10 years. If the policy is purchased by an individual aged 37, for whom the life table below applies, and the annual interest rate is $i = 0.065$, what is the expected present value of this annuity?

x	l_x	d_x
37	10000.00	4.94
38	9995.06	5.17
39	9989.89	5.42
40	9984.47	5.69
41	9978.77	5.99
42	9972.79	6.31

x	l_x	d_x
43	9966.47	6.66
44	9959.81	7.05
45	9952.76	7.47
46	9945.29	7.92
47	9937.37	8.42

Formulae for Present Value of a Whole-Life Continuous Annuity

$$\bar{a}_x = \frac{1 - \bar{A}_x}{\delta}$$

$$\bar{a}_x = \int_0^\infty e^{-\delta t}\,{}_tp_x\,dt$$

$$\bar{a}_x = \int_0^\infty \bar{a}_{\overline{t}|}\,{}_tp_x\mu_{x+t}\,dt$$

Variance of present value is given by:

$$\text{Var}(Y) = \frac{{}^2\bar{A}_x - (\bar{A}_x)^2}{d^2}$$

Question 34

Suppose lifetime is modelled as

$$F_0(t) = 1 - \left(1 - \frac{x}{130}\right)^{\frac{1}{4}}$$

Using each of the formulae on the previous slide, calculate the EPV of a continuous annuity at a rate of \$1 per year, on a life currently aged 42, if the annual force of interest is given by $\delta = 0.04$.

Question 35

Using a certain life table, we have that $A_{65}^{(12)} = 0.5689$, and the current annual interest rate is $i^{(12)} = 6\%$. A man to whom this lifetable applies has saved up $857,000 for his retirement, and wishes to purchase a whole-life annuity with monthly payments. How much should the monthly payments be?

Question 36

(a) Order the following from smallest to largest:

- $\bar{a}_x$
- a_x
- $\ddot{a}_x$
- $a_x^{(m)}$
- $\ddot{a}_x^{(m)}$

Why are they always in this order?

(b) Does the difference between the different payment frequencies increase or decrease as x increases?

Question 37

Mr. Allen is currently aged 49. He has just inherited $800,000, and he plans to invest it in a deferred annuity, with annual payments starting when he turns 65. If the interest rate is $i = 7\%$ and the appropriate life table is as given on the next slide, what should the annual payments be in the deferred annuity?

Life Table for Question 37

x	l_x	d_x	x	l_x	d_x	x	l_x	d_x
49	10000.00	40.81	66	8257.75	216.46	83	2190.69	385.30
50	9959.19	45.19	67	8041.29	235.64	84	1805.39	355.27
51	9914.00	50.06	68	7805.65	255.73	85	1450.12	319.28
52	9863.94	55.45	69	7549.92	276.57	86	1130.84	278.58
53	9808.49	61.42	70	7273.35	297.94	87	852.26	234.91
54	9747.08	68.02	71	6975.42	319.54	88	617.35	190.39
55	9679.06	75.31	72	6655.88	340.99	89	426.96	147.33
56	9603.75	83.34	73	6314.89	361.84	90	279.63	107.97
57	9520.41	92.17	74	5953.05	381.53	91	171.66	74.16
58	9428.24	101.87	75	5571.52	399.41	92	97.50	47.13
59	9326.37	112.50	76	5172.11	414.75	93	50.37	27.24
60	9213.87	124.10	77	4757.36	426.75	94	23.12	14.00
61	9089.76	136.74	78	4330.60	434.58	95	9.13	6.18
62	8953.02	150.46	79	3896.03	437.38	96	2.95	2.23
63	8802.56	165.28	80	3458.64	434.39	97	0.71	0.61
64	8637.28	181.23	81	3024.26	424.94	98	0.11	0.10
65	8456.05	198.30	82	2599.31	408.62	99	0.01	0.01

Question 38

An indivdual aged 65 is planning to purchase a life annuity. She is told that the cost of an annuity that pays $800 per month is $120,000. She is also told that the cost for the same annuity for someone 75 years old is $97,000. If the current rate of interest is $i^{(12)} = 4\%$ and the individuals probability of surviving to age 75 is 0.87, what is the cost of a life annuity that pays $800 per month and is guaranteed for 10 years?

Question 39

A particular disability insurance policy pays \$10000 at the start of the first year of disability, and increases the annual payment by \$1000 every year thereafter for a maximum of 10 years. An individual aged 52 has just become disabled and is starting to claim benefits under this policy. The life table for this individual is as shown below. What is the EPV of the benefits payed out under this policy if the current interest rate is $i = 6\%$?

x	l_x	d_x
52	10000.00	787.38
53	9212.62	811.55
54	8401.07	827.99
55	7573.08	835.08
56	6738.01	831.30
57	5906.71	815.36

x	l_x	d_x
58	5091.35	786.36
59	4304.99	743.96
60	3561.04	688.56
61	2872.48	621.47
62	2251.01	544.93

Question 40

A woman aged 76 wants to purchase an annual life annuity to fund her retirement. She wants the annuity to pay \$15,000 in the first year and increase by 4% each year. The current interest rate is $i = 7\%$, and the life table below is appropriate for this woman. How much should she pay for this annuity?

x	l_x	d_x
76	10000.00	613.73
77	9386.27	638.51
78	8747.75	659.61
79	8088.14	676.05
80	7412.08	686.80
81	6725.28	690.84
82	6034.45	687.22
83	5347.23	675.13
84	4672.10	654.02
85	4018.08	623.62

x	l_x	d_x
86	3394.46	584.13
87	2810.32	536.22
88	2274.10	481.12
89	1792.98	420.61
90	1372.36	356.98
91	1015.38	292.88
92	722.51	231.09
93	491.42	174.29
94	317.13	124.72
95	192.41	83.91

x	l_x	d_x
96	108.49	52.47
97	56.02	30.05
98	25.98	15.45
99	10.53	6.94
100	3.58	2.62
101	0.96	0.78
102	0.18	0.16
103	0.02	0.02

Uniform Distribution of Deaths

- Recall that under UDD we have $A_x^{(m)} = \frac{i}{i^{(m)}} A_x$.
- We can combine this with the formula $\ddot{a}_x = \frac{1-A_x}{d}$ to get

$$\begin{aligned}
\ddot{a}_x^{(m)} &= \frac{1 - A_x^{(m)}}{d^{(m)}} \\
&= \frac{1 - \frac{i}{i^{(m)}} A_x}{d^{(m)}} \\
&= \frac{i}{i^{(m)}} \frac{1 - A_x}{d^{(m)}} - \frac{\frac{i}{i^{(m)}} - 1}{d^{(m)}} \\
&= \frac{id}{i^{(m)} d^{(m)}} \frac{1 - A_x}{d} - \frac{i - i^{(m)}}{i^{(m)} d^{(m)}} \\
&= \frac{id}{i^{(m)} d^{(m)}} \ddot{a}_x - \frac{i - i^{(m)}}{i^{(m)} d^{(m)}}
\end{aligned}$$

Question 41

The price of an annual life annuity-due that pays $12,000 a year to a life aged 65 is $112,000. If the current interest rate is $i^{(12)} = 0.06$, what is the price for a monthly annuity-due that pays $1,000 a month to the same life aged 65?

Question 42

The EPV of an annual life annuity-due that pays $20,000 a year to a life aged 65 is $336,000. The EPV of a monthly annuity-due that pays $2,000 a month to a life aged 65 is $392,100. If this is based on the UDD assumption, what rate of interest $i^{(12)}$ is being applied to calculate the EPV?

(a) 0.0222 (b) 0.0354 (c) 0.0442 (d) 0.0590

5.11 Evaluating Annuity Functions

Woolhouse's formula

Euler-Maclaurin formula:

$$\int_0^\infty g(t)dt = h\sum_{k=0}^{\infty} g(kh) - \frac{h}{2}g(0) + \frac{h^2}{12}g'(0) - \frac{h^4}{720}g'''(0) + \cdots$$

Continuous case:

$$\bar{a}_x = \ddot{a}_x - \frac{1}{2} - \frac{1}{12}(\delta + \mu_x)$$

Discrete case:

$$\ddot{a}_x^{(m)} = \ddot{a}_x - \frac{m-1}{2m} - \frac{m^2-1}{12m^2}(\delta + \mu_x)$$

If μ_x is unknown, we often use $\frac{q_{x-1}+q_x}{2}$ as an approximation.

Question 43

A certain insurance policy involves a term annuity with a monthly payment in advance of $100 for a term of 10 years. Suppose the policy is purchased by an individual aged 37, for whom the life table below applies and the current effective interest rate is 6.5%.

x	l_x	d_x
36	10000.00	4.73
37	9995.27	4.94
38	9990.33	5.17
39	9985.16	5.42
40	9979.74	5.69
41	9974.05	5.99

x	l_x	d_x
42	9968.07	6.31
43	9961.76	6.66
44	9955.10	7.05
45	9948.05	7.46
46	9940.59	7.92
47	9932.67	8.42

Recall from Question 33 that the EPV of an equivalent policy with annual $1,200 payments is $9,166.78. Use Woolhouse's formula to calculate the expected present value of this annuity.

Question 44

Suppose lifetime is modelled as

$$F_0(t) = 1 - \left(1 - \frac{t}{130}\right)^{\frac{1}{4}}$$

Calculate $a_x^{(12)}$ for $i^{(12)} = 0.04$, for $x = 20, 30, 40, 50, 60, 70, 80, 90, 100$ using:

(a) Summing the series explicitly

(b) Using the UDD assumption

(c) Using the first 2 terms of Woolhouse's formula

(d) Using the first 3 terms of Woolhouse's formula

(e) Using the first 3 terms of Woolhouse's formula with μ_x approximated by $\frac{q_{x-1}+q_x}{2}$.

Solution to Question 44

x	A_x	Exact	UDD	W2	W3	W3*
20	23.5646	23.1040	23.1027	23.1063	23.1028	23.1028
30	23.2618	22.8017	22.7998	22.8034	22.7999	22.7999
40	22.8695	22.4104	22.4075	22.4112	22.4076	22.4076
50	22.3541	21.8965	21.8921	21.8958	21.8922	21.8922
60	21.6678	21.2123	21.2056	21.2094	21.2058	21.2058
70	20.7422	20.2902	20.2799	20.2838	20.2802	20.2802
80	19.4796	19.0333	19.0172	19.0213	19.0176	19.0176
90	17.7397	17.3024	17.2771	17.2814	17.2776	17.2776
100	15.3197	14.8971	14.8567	14.8613	14.8573	14.8573

Question 45

The current interest rate is $i = 0.07$. The lifetable for select lives is

x	$l_{[x]}$	$l_{[x]+1}$	$l_{[x]+2}$	$l_{[x]+3}$
41	99824.07	99787.27	99737.35	99669.47
42	99738.86	99700.43	99648.19	99577.01
43	99649.63	99609.43	99554.64	99479.82
44	99556.01	99513.84	99456.25	99377.45
45	99457.54	99413.23	99352.57	99269.38
46	99353.77	99307.09	99243.07	99155.08
47	99244.17	99194.90	99127.18	99033.92
48	99128.18	99076.07	99004.29	98905.24
49	99005.17	98949.94	98873.71	98768.31
50	98874.46	98815.81	98734.70	98622.34
51	98735.31	98672.91	98586.44	98466.45

Calculate the EPV of a ten-year life annuity with annual payments of \$30,000, made to an individual aged 42, who was select one year ago.

Question 46

An insurance policy assumes the following lifetable for an individual currently aged 52:

x	l_x	d_x
52	10000.00	229.92
53	9770.08	262.48
54	9507.60	298.55
55	9209.04	338.09
56	8870.95	380.86
57	8490.09	426.34

x	l_x	d_x
58	8063.75	473.71
59	7590.04	521.68
60	7068.35	568.49
61	6499.86	611.79
62	5888.08	648.63

The policy charges a premium of $30,000 per year for 10 years, and pays a benefit of $500,000 in the event of death of the insured. If the interest rate is $i = 0.04$, what is the probability that the net future loss on this policy exceeds $250,000?

Question 47

For the policy from Question 46, (see lifetable below) which pays \$500,000 in the event of death of the insured, what annual premium should be charged using the equivalence principle and an interest rate of $i = 0.04$?

x	l_x	d_x
52	10000.00	229.92
53	9770.08	262.48
54	9507.60	298.55
55	9209.04	338.09
56	8870.95	380.86
57	8490.09	426.34

x	l_x	d_x
58	8063.75	473.71
59	7590.04	521.68
60	7068.35	568.49
61	6499.86	611.79
62	5888.08	648.63

Question 48

A life aged 55, to whom the lifetable below applies, wants to purchase a deferred annuity that will pay $24,000 a year, starting in the year when she turns 65. Using the appropriate lifetable, the pensions company determines that the expected payment for an immediate annuity with annual payment $24,000 for a life aged 65, using the same lifetable, is $340,000. What annual premium should be charged for the policy if the appropriate interest rate is $i = 0.04$?

x	l_x	d_x
55	10000.00	9.51
56	9990.49	10.00
57	9980.49	10.53
58	9969.95	11.11
59	9958.85	11.73
60	9947.12	12.40

x	l_x	d_x
61	9934.72	13.12
62	9921.60	13.90
63	9907.70	14.74
64	9892.96	15.65
65	9877.31	16.63

Question 49

The interest rate is $i = 0.06$. A whole life insurance contract for an individual aged 64 with death benefits of \$1,200,000 has annual premiums of \$86,000 payable until age 80. What premium should be charged to a select individual aged 58 for a whole life insurance policy with the same benefit and premiums payable until age 80. Use the table below. You are also given that the probability of an individual aged 64 surviving to 80 is 0.582314, and that $A_{80} = 0.7731$.

x	$l_{[x]}$	$l_{[x]+1}$	$l_{[x]+2}$	$l_{[x]+3}$
58	99752.79	99618.74	99442.26	99209.76
59	99489.46	99344.93	99154.47	98903.36
60	99205.23	99049.28	98843.58	98572.18
61	98898.23	98729.81	98507.52	98214.03
62	98566.39	98384.41	98144.05	97826.50
63	98207.51	98010.76	97750.72	97407.01
64	97819.18	97606.36	97324.94	96952.78

Question 50

A life aged 55 wants to purchase a term insurance policy for a term of 10 years. The benefits should be \$800,000. For a policy with annual premiums, the annual premium is \$2,300. The probability that the life dies within the 10 years is 0.0338. If he wants to pay monthly premiums, and benefits to be paid at the end of the month of death, what should the monthly premium be? Assume interest rates are $i^{(12)} = 0.06$, and his current mortality is given by $\mu_{55} = 0.0024$ and $\mu_{55} = 0.0048$. Calculate the premium using:

(a) Uniform distribution of deaths

(b) Woolhouse's formula

Question 51

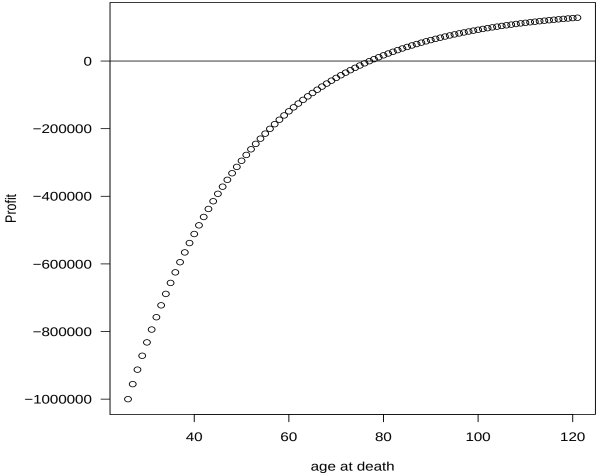
An insurer issues a whole life insurance contract with a death benefit of \$700,000 to a select individual aged 41. Net monthly premiums, payable until age 80 are \$1,380. You are given that $a^{(12)}_{41:\overline{39}|} = 14.32$. The interest rate is $i^{(12)} = 0.05$. The insurer has initial costs of \$6,000 plus 50% of the first premium, and renewal costs of 3% of each subsequent premium. What should the gross monthly premium be?

Question 52

A select individual aged 34 wants to purchase endowment insurance for a term of 20 years. The insurance will pay $600,000 in the event of the individual's death, or at the end of the 20 years. The net premium for this insurance is $2,300 a month. The current interest rate is $i^{(12)} = 0.07$. The insurance company has initial costs of $1,500 plus 70% of the first premium and monthly renewal costs of 0.01% of the insured benfit. Calculate the gross premium.

6.7 Profit

Figure: Net profit at time of starting policy on a whole life insurance policy with death benefit \$1,000,000, annual premium \$6,000, interest rate $i = 0.04$, as a function of age at death.



Question 53

A whole life insurance policy to an individual aged 54 has a death benefit of \$700,000 and an annual premium of \$28,000. If interest is $i = 0.07$ and the life table for this life is as below, what is the probability that the policy makes a net profit?

x	l_x	d_x
54	10000.00	18.07
55	9981.93	19.80
56	9962.13	21.70
57	9940.43	23.77
58	9916.66	26.04
59	9890.63	28.52
60	9862.11	31.23
61	9830.88	34.19

x	l_x	d_x
62	9796.70	37.42
63	9759.28	40.94
64	9718.33	44.79
65	9673.55	48.97
66	9624.58	53.53
67	9571.05	58.48
68	9512.57	63.85
69	9448.72	69.68

Question 54

An insurer issues 10-year term insurance to 200 lives aged 43. The policies have a death benefit of $1,000,000. The initial costs are $1,000 plus 40% of the first premium. The renewal costs are 1.5% of each subsequent premium. The following life table is to be used:

x	l_x	d_x
43	10000.00	8.64
44	9991.36	9.55
45	9981.80	10.58
46	9971.23	11.73
47	9959.50	13.01
48	9946.49	14.45

x	l_x	d_x
49	9932.04	16.07
50	9915.97	17.87
51	9898.10	19.89
52	9878.20	22.16
53	9856.04	24.69

If interest is $i^{(12)} = 0.06$ and the UDD assumption is to be used, calculate the premium which ensures a 95% probability that the policies will result in a profit.

6.9 Extra Risks

Common Approaches to Price Policies for Impaired Lives

- Treat the life as k years older than they really are. Simplest approach as the rates already calculated.
- Increase mortality μ_x by a constant c. Recall that ${}_tp_x = e^{-\int_0^t \mu_{x+s}\,ds}$, so for increased mortality we have

$$ {}_tp_x = e^{-\int_0^t \mu_{x+s}+c\,ds} = e^{-\int_0^t \mu_{x+s}\,ds - ct} = p_x e^{-ct} $$

$$ \bar{a}_x = \int_{t=0}^{\infty} e^{-\delta t}\,{}_tp_x e^{-ct}\,dt = \int_{t=0}^{\infty} e^{-(\delta+c)t}\,{}_tp_x\,dt $$

  which is the annuity function for a standard life at a higher force of interest. The benefit A_x can be recalculated from this. (This also works for term annuities and annual annuities.)
- Increase mortality μ_x by a constant factor. This involves recalculating all quantities. For a Gompertz model, it is the same as treating the life as older.

Question 55

The interest rate is 0.04. The standard lifetable for women is:

x	l_x	d_x	x	l_x	d_x	x	l_x	d_x
36	10000.00	2.76	42	9979.48	4.52	48	9945.73	7.47
37	9997.24	3.00	43	9974.96	4.91	49	9938.26	8.13
38	9994.24	3.25	44	9970.05	5.34	50	9930.13	8.85
39	9990.99	3.53	45	9964.72	5.80	51	9921.27	9.64
40	9987.47	3.83	46	9958.91	6.31	52	9911.63	10.49
41	9983.64	4.16	47	9952.60	6.87	53	9901.14	11.43

Calculate the annual premium for a 10-year term insurance with death benefit $1,000,000 on:

(a) a standard life aged 36.

(b) an impaired life aged 36, who is treated as being 5 years older.

(c) a life aged 36, who works in a hazardous environment which increases her mortality rate by 0.006.

(d) a life aged 36 whose mortality is 1.2 times the normal mortality.

Table: Annual Premiums for whole life insurance with death benefits \$100,000, premiums payable until death, interest rate $i = 0.04$, mortality following Makeham's model with $A = 0.0000708$, $B = 0.00001044$, $C = 1.121$, under various adjustments for extra risks.

age	Standard Premium	age+5	mortality+0.006	mortality×1.2
30	771.10	969.68	1215.18	829.91
40	1228.44	1569.95	1651.40	1328.00
50	2027.53	2651.41	2430.55	2206.41
60	3519.47	4755.59	3906.35	3867.22
70	6562.38	9280.27	6938.24	7310.16
80	13495.27	20236.19	13858.15	15306.74
90	31334.00	50068.37	31642.40	36271.61
100	81856.71	—	81931.27	96153.85

7. Policy Values

Policy Values

- For a typical life insurance policy, expected benefits are higher in later years, while premiums remain constant.
- This means that in earlier years, the premiums are too much for the benefits, while in later years the premiums are not sufficient to cover the benefits.
- This means that the insurance company must set some reserves aside to cover the expected benefits of the policy. The amount of reserves needed is called the policy value.

Question 56

The life table for standard lives is

x	l_x	d_x
37	10000.00	4.52
38	9995.48	4.90
39	9990.58	5.32
40	9985.25	5.78
41	9979.47	6.29
42	9973.18	6.85

x	l_x	d_x
43	9966.33	7.46
44	9958.86	8.14
45	9950.72	8.88
46	9941.84	9.70
47	9932.14	10.60

The interest rate is $i = 0.08$. The resulting annual net premium for a 10-year term insurance with a death benefit of $200,000 on a life aged 37 is therefore $119.32. Calculate the policy value at each year of the term of the policy.

Table: Policy values for a 10-year life insurance policy on a life aged 37 for each year of the policy

age	a_x	A_x	Premium	Policy Value
37	7.23	862.82	119.32	0.00
38	6.73	841.83	125.04	38.49
39	6.19	811.53	131.01	72.42
40	5.61	770.36	137.25	100.64
41	4.98	716.64	143.77	121.85
42	4.31	648.32	150.56	134.50
43	3.57	563.20	157.62	136.85
44	2.78	458.90	165.01	127.06
45	1.93	332.41	172.67	102.70
46	1.00	180.68	180.68	61.36

Comments on Policy Values

- Policy value basis may have changed since the policy was introduced.
- Net premium policy value uses a premium based on the new policy value basis.
- Gross premium policy value uses actual contract premium.
- If gross policy value at time 0 is positive, it means the valuation basis is more conservative than the basis used to calculate the premium. If it is negative, it means the premium basis is more conservative.
- Note how the policy value increases over time. The early premiums more than cover the risks in that year. The surplus is necessary to cover expected losses in future years.

Question 57

Repeat Question 56 using the select life table below. [Assume the life is select at 37].

x	$l_{[x]}$	$l_{[x]+1}$	$l_{[x]+2}$	$l_{[x]+3}$
37	9995.86	9993.18	9989.73	9985.25
38	9990.99	9988.08	9984.33	9979.47
39	9985.70	9982.54	9978.46	9973.18
40	9979.95	9976.52	9972.08	9966.33
41	9973.69	9969.97	9965.13	9958.86
42	9966.88	9962.83	9957.56	9950.72
43	9959.46	9955.04	9949.30	9941.84
44	9951.37	9946.55	9940.29	9932.14
45	9942.55	9937.29	9930.45	9921.55
46	9932.91	9927.16	9919.69	9909.97
47	9922.38	9916.10	9907.94	9897.30

Table: Policy values for a 10-year life insurance policy on a select life aged 37 for each year of the policy

age	a_x	A_x	Premium	Policy Value
37	7.23	790.99	109.35	0.00
38	6.73	800.87	118.93	64.50
39	6.19	796.16	128.52	118.75
40	5.61	770.51	137.28	156.73
41	4.98	716.79	143.80	171.70
42	4.31	648.49	150.60	177.59
43	3.57	563.39	157.67	172.65
44	2.78	458.90	165.01	154.78
45	1.93	332.41	172.67	121.89
46	1.00	180.68	180.68	71.33

Question 58

A life aged 44 purchases a 10-year endowment policy, which pays a benefit of \$1,000,000 either at the end of 10 years, or at the end of the year of death. The appropriate life table is given below. The annual premiums are calculated as \$72,111.08.

x	l_x	d_x
44	10000.00	8.08
45	9991.92	8.84
46	9983.08	9.68
47	9973.39	10.61
48	9962.78	11.64
49	9951.14	12.77

x	l_x	d_x
50	9938.37	14.02
51	9924.35	15.40
52	9908.95	16.92
53	9892.03	18.59
54	9873.44	20.44

Using a policy value basis of interest rate 0.05%:

(a) calculate the gross policy value at time 0.

(b) calculate the gross and net policy values at time 5.

Question 59

An insurance policy is paid to a life aged 55. The life follows the life table below.

x	l_x	d_x
55	10000.00	23.94
56	9976.06	26.35
57	9949.71	29.01
58	9920.70	31.93
59	9888.77	35.15
60	9853.62	38.68

x	l_x	d_x
61	9814.94	42.57
62	9772.38	46.83
63	9725.54	51.51
64	9674.04	56.63
65	9617.40	62.24

The policy pays a benefit of $600,000 if the life survives to age 65. If the life dies before age 65, at the end of the year of death, the policy pays a benefit equal to the policy value at the start of the year in which the life dies. If the annual premium is $54,000 and the interest rate is $i = 0.07$, calculate the policy value after 3 years.

Question 60

Recall Question 56, where the annual net premium for a 10-year term insurance with a death benefit of \$200,000 on a life aged 37 was calculated as \$119.32. The policy value after 1 year was calculated as \$38.49, based on the lifetable with $q_{37} = 0.000452$ and an interest rate $i = 0.08$.

Suppose the company sells 1,200 such policies in a given year, and in the first year of the policies:

- one policy holder dies
- the company earns an interest rate of $i = 0.095$
- the company needs to pay expenses of \$1,000 related to the policies at the end of the year.

What is the company's profit or loss for the year on these policies?

Question 61

Recall Question 56, where the annual net premium for a 10-year term insurance with a death benefit of \$200,000 on a life aged 37 was calculated as \$119.32.
Suppose the company sells 1,200 such policies in a given year, and in the first three years of the policies:

- one policyholder dies in the first year.
- the company earns an annual interest rate of $i = 0.085$ in the first year, and $i = 0.09$ over the remaining two years.
- the company needs to pay expenses related to the policies of \$1,000 at the end of each year.

Calculate the asset share of the remaining policies.

Question 62

As for Question 58, $i = 0.05$ and the lifetable for a life aged 44 is

x	l_x	d_x
44	10000.00	8.08
45	9991.92	8.84
46	9983.08	9.68
47	9973.39	10.61
48	9962.78	11.64
49	9951.14	12.77

x	l_x	d_x
50	9938.37	14.02
51	9924.35	15.40
52	9908.95	16.92
53	9892.03	18.59
54	9873.44	20.44

A life aged 44 purchases a 10-year endowment policy, which pays a benefit of \$1,000,000 either at the end of 10 years, or at the end of the month of death. For annual premiums, we have $A_{44:\overline{10}|} = 0.61556$ and $A_{49:\overline{5}|} = 0.784108$. Using Woolhouse's formula:

(a) calculate the monthly premium.

(b) calculate the policy value after 5 years 2 months.

Question 63

As for Question 58, $i = 0.05$ and the lifetable for a life aged 44 is

x	l_x	d_x
44	10000.00	8.08
45	9991.92	8.84
46	9983.08	9.68
47	9973.39	10.61
48	9962.78	11.64
49	9951.14	12.77

x	l_x	d_x
50	9938.37	14.02
51	9924.35	15.40
52	9908.95	16.92
53	9892.03	18.59
54	9873.44	20.44

A life aged 44 purchases a 10-year endowment policy, which pays a benefit of \$1,000,000 either at the end of 10 years, or at the end of the month of death. Recall from the previous question that the policy value after 5 years 2 months is \$455,054.36.

Calculate the policy value after 5 years 1.6 months.

Recurrence for Annual Policy Values

$$
\begin{aligned}
{}_tV &= SA_{x+t} - P\ddot{a}_{x+t} \\
&= SA_{x+t} - P\frac{1-A_{x+t}}{d} \\
&= \left(S + \frac{P}{d}\right) A_{x+t} - \frac{P}{d} \\
&= \left(S + \frac{P}{d}\right) \left((1+i)^{-1}(q_{x+t} + p_{x+t}A_{x+t+1})\right) - \frac{P}{d} \\
&= (1+i)^{-1} p_{x+t} \left(S + \frac{P}{d}\right) A_{x+t+1} + \left(S + \frac{P}{d}\right) (1+i)^{-1} q_{x+t} - \frac{P}{d} \\
&= (1+i)^{-1} p_{x+t} \left({}_{t+1}V + \frac{P}{d}\right) + \left(S + \frac{P}{d}\right) (1+i)^{-1} q_{x+t} - \frac{P}{d} \\
&= (1+i)^{-1} \left(p_{x+t}\,{}_{t+1}V + Sq_{x+t}\right) - P
\end{aligned}
$$

Thiele's differential equation

$$\frac{d}{dt}\,{}_tV = \delta_t\,{}_tV + P_t - (S_t - {}_tV)\mu_{x+t}$$

where:

- ${}_tV$ is the policy value at time t if the policy is still in force.
- δ_t is the force of interest at time t.
- P_t is the rate of premium payment at time t (minus expenses).
- S_t is the death benefit at time t (plus expenses).
- μ_{x+t} is the force of mortality at time t.

Question 64

Suppose Mortality is given by Makeham's law

$$\mu_x = 0.000084 + 0.0000104(1.099)^x$$

Use a numerical solution to Thiele's differential equation to calculate the policy value of a 10-year term insurance with death benefit \$500,000 and force of interest given by $\delta = 0.045$, sold to a life aged 45, with the appropriate net premium (paid continuously) for this policy.

Answer to Question 64

h	0.1	0.01	0.001	0.0001	0.00005
Premium=	569.63	620.10	623.34	625.66	625.69
age=45	0	0	0	0	0
age=46	151.09	201.40	204.59	206.95	206.98
age=47	270.91	373.65	380.16	385.00	385.05
age=48	354.20	511.59	521.55	528.98	529.07
age=49	395.09	609.44	622.99	633.12	633.24
age=50	387.01	660.72	677.99	690.95	691.11
age=51	322.62	658.15	679.29	695.22	695.41
age=52	193.69	593.62	618.78	637.81	638.04
age=53	-8.96	458.05	487.37	509.65	509.91
age=54	-295.58	241.28	274.92	300.60	300.90
age=55	-677.63	-68.05	-29.93	-0.68	-0.34

7.6 Policy Alterations

Common Policy Alterations

- Surrender – cancel policy with immediate effect, and may receive a surrender value.
- Stop paying premiums and receive a reduced sum insured.
- Convert from whole-life to paid-up term

Reasons for Penalising Alterations

- Adverse selection
- Expenses in changing policy
- Liquidity risk

Question 65

Six years ago a woman then aged 37 purchased whole life insurance for a death benefit of $400,000. The appropriate lifetable is

x	l_x	d_x	x	l_x	d_x	x	l_x	d_x
37	10000.00	4.11	43	9969.60	6.70	49	9919.02	11.37
38	9995.89	4.44	44	9962.89	7.31	50	9907.65	12.45
39	9991.45	4.81	45	9955.59	7.97	51	9895.21	13.63
40	9986.64	5.22	46	9947.62	8.70	52	9881.57	14.94
41	9981.42	5.67	47	9938.92	9.50	53	9866.63	16.38
42	9975.76	6.16	48	9929.42	10.39	54	9850.25	17.96

The annual premium is $2002.57 ($i = 0.05$). The policy has a cash surrender value of 85% of the policy value minus $150.

(a) Calculate the cash surrender value of the policy.

(b) If the woman wishes to change the insurance to endowment insurance, which will pay a benefit of $100,000 either when she turns 55 or at death, what should the new premiums be?

Question 66

Four years ago, a man aged 53 took out a term insurance policy with a term of 10 years and a death benefit of \$250,000. The insurance company estimated that the probability of his dying during the term of the insurance was 0.021, and was calculated the monthly net premium as \$54.71. If the man wanted to take out the same term insurance today for the remaining 6 years, the monthly premium would be \$61.17, and his chance of dying during the term would be 0.018. The interest rate is $i = 0.04$. The policy has a cash surrender value of 85% of the policy value.

(a) If the man wants to convert the policy into a paid-up policy for the remainder of the term (i.e. the next six years), what are the new death benefits?

(b) If the man wants to reduce his monthly premiums to \$25, what are the new death benefits?

Question 67

A man aged 44 buys a whole life insurance policy. The annual premiums are \$600 for the first 4 years, and \$1,000 after then until age 80. The death benefits are \$80,000 for the first 3 years, \$100,000 for the next 4 years and \$400,000 from then on. The interest rate is $i = 0.07$ and the appropriate lifetable is:

x	l_x	d_x
44	10000.00	3.45
45	9996.55	3.72
46	9992.83	4.02
47	9988.82	4.35

x	l_x	d_x
48	9984.47	4.71
49	9979.75	5.12
50	9974.64	5.57
51	9969.07	6.06

Calculate the retrospective policy value after 2 years.

7.9 Deferred Acquisition Expenses and Modified Premium Reserves Payments

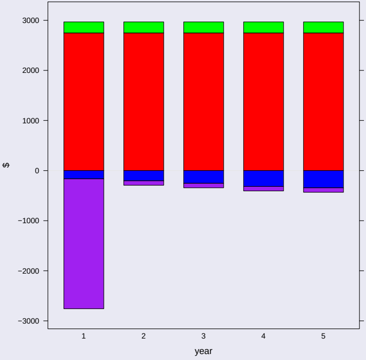
Breakdown of Gross Premiums and Reserves

- Gross premium can be divided as sum of net premium plus expense premium
- Similarly, we can divide the Gross reserve into a net reserve and an expense reserve.
- Net premiums are larger than expected benefits in early years, smaller in later years.
- Expense premiums are smaller than expenses in early years, larger in later years.
- This means expense reserves are negative! Net reserves are too large.

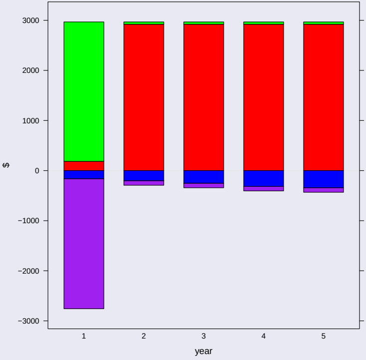
7.9 Deferred Acquisition Expenses and Modified Premium Reserves Payments

Breakdown of Gross Premiums

(a) Without FPT



(b) With FPT



Question 68

A man aged 29 buys a whole-life insurance policy with death benefits of \$300,000. The lifetable for this man is:

x	l_x	d_x
29	10000.00	2.28
30	9997.72	2.40
31	9995.33	2.53
32	9992.80	2.68

The interest rate is $i = 0.06$.
You are given that $A_{29} = 0.0343763$, $A_{30} = 0.0362191$, $A_{31} = 0.0381614$, $A_{32} = 0.0402081$. Using the lifetable above, and using a Full Preliminary Term of 1 year, calculate the policy value after 2 years.