ACSC/STAT 3720, Life Contingencies I
Winter 2018
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Homework Sheet 4
Model Solutions

## Basic Questions

1. *Using the lifetable in Table 1, calculate $\ddot{a}_{[43]+3}$ at interest rate $i = 0.05$. You are given that $A_{[43]+3} = 0.123339$.*

   Using the formula for conversion, we get

$$\ddot{a}_{[43]+3} = (1 - A_{[43]+3})\frac{1.05}{0.05} = 21 \times (1 - 0.123339) = 18.409881$$

2. *An individual aged 39 for whom Table 1 is appropriate, takes out a 5-year term insurance policy. The annual premiums are \$728, payable at the begining of each year. If the current interest rate is $i = 0.04$, what is the expected present value of the premiums paid?*

   We get

$$\ddot{a}_{39:\overline{5}|} = 1+\frac{9958.44}{9962.82}(1.04)^{-1}+\frac{9953.69}{9962.82}(1.04)^{-2}+\frac{9948.55}{9962.82}(1.04)^{-3}+\frac{9942.98}{9962.82}(1.04)^{-4} = 4.62564963618$$

   so the EPV is $4.62564963618 \times 728 = \$3,367.47$.

3. *An annuity pays out continuously at a rate of \$5,000 a year until the death of an individual currently aged 68 to whom the ultimate part of Table 1 applies. What is the expected present value of this annuity, using the uniform distribution of deaths assumption, and force of interest $\delta = 0.05$? You are given the following values of $A_{68}$ at various interest rates:*

| $i$ | $A_{68}$ |
|---|---|
| 0.03922071 | 0.380644 |
| 0.04 | 0.374228 |
| 0.040811 | 0.367698 |
| 0.04879016 | 0.310707 |
| 0.05 | 0.303101 |
| 0.0512711 | 0.295372 |

   Force of interest $\delta = 0.05$ gives $i = e^{0.05} - 1 = 0.05127109638$, so we get $A_{68} = 0.295372$. We now use UDD to get

$$\overline{A}_{68} = \frac{i}{\delta}A_{68} = \frac{0.05127109638}{0.05} \times 0.295372 = 0.302880925599$$

   Using our standard conversion formula, we get

$$\overline{a}_{68} = \frac{1 - \overline{A}_{68}}{\delta} = \frac{1 - 0.302880925599}{0.05} = 13.942381488$$

   The EPV of the annuity is therefore $5000 \times 13.942381488 = \$69,711.91$.

4. *A pension plan pays monthly benefits of $4,000 to an individual aged 68. What is the expected present value of the benefit under the uniform distribution of deaths assumption, interest rate $i^{(12)} = 0.04$ and the lifetable in Table 1? [These allow us to calculate $A_{68} = 0.368252$.]*

   Using UDD, we get

$$A_{68}^{(12)} = \frac{i}{i^{(12)}} A_{68} = \frac{1.0033333333^{12} - 1}{0.04} \times 0.368252 = 0.37507886281$$

   Now the conversion formula gives

$$\ddot{a}_{68}^{(12)} = (1 - 0.37507886281)\frac{1.003333333333}{0.04} = 15.6751051912$$

   The EPV of the benefit is therefore $4000 \times 12 \times 15.6751051912 = \$752,405.05$.

## Standard Questions

5. *An individual aged 67 receives an annual pension of $40,000 at the start of each year. The pension is guaranteed for 5 years. She wants to increase the guaranteed period to 10 years, but keep the EPV of payments the same. The interest rate is $i = 0.06$, and the ultimate part of the lifetable in Table 1 applies to this individual. Calculate the new payments. [You calculate $\ddot{a}_{67} = 13.4578836667$, $\ddot{a}_{72} = 12.4660416667$, and $\ddot{a}_{77} = 11.3298806667$.]*

   The current EPV is

$$\begin{aligned}
&40000\left(\ddot{a}_{\overline{5}|0.06} + {}_5p_{67}(1.06)^{-5}\ddot{a}_{72}\right)\\
&\qquad = 40000\left(\frac{1.06 - 1.06^{-4}}{0.06} + \frac{9203.55}{9485.52}(1.06)^{-5} \times 12.4660416667\right)\\
&\qquad = \$540,141.82
\end{aligned}$$

   For an annuity of $1 per year, guaranteed for 10 years, the EPV is

$$\begin{aligned}
&\ddot{a}_{\overline{10}|0.06} + {}_{10}p_{67}(1.06)^{-10}\ddot{a}_{77}\\
&\qquad = \frac{1.06 - 1.06^{-9}}{0.06} + \frac{8775.52}{9485.52}(1.06)^{-10} \times 11.3298806667\\
&\qquad = 13.6546905917
\end{aligned}$$

   The new annuity payments are therefore

$$\frac{540141.82}{13.6546905917} = \$39,557.24$$

6. *A man aged 106, to whom the ultimate part of the lifetable in Table 1 applies, wants a pension which will pay $20,000 in a year's time, and thereafter will provide annual payments increasing by 3% every year (so the second payment when the man turns 108 will be $20,600).*

*What is the expected present value of the benefits of this pension if the current interest rate is $i = 0.05$?*

We calculate the "real" rate of interest is $i^* = \frac{0.05-0.03}{1.03} = 0.0194174757282$. At this rate of interest, we calculate

$$\begin{aligned}
a_{106} = &\frac{1215.44}{1474.18}(1.0194174757282)^{-1} + \frac{981.65}{1474.18}(1.0194174757282)^{-2} + \frac{774.71}{1474.18}(1.0194174757282)^{-3} \\
&+ \frac{595.71}{1474.18}(1.0194174757282)^{-4} + \frac{444.87}{1474.18}(1.0194174757282)^{-5} + \frac{321.41}{1474.18}(1.0194174757282)^{-6} \\
&+ \frac{223.65}{1474.18}(1.0194174757282)^{-7} + \frac{149.10}{1474.18}(1.0194174757282)^{-8} + \frac{94.62}{1474.18}(1.0194174757282)^{-9} \\
&+ \frac{56.74}{1474.18}(1.0194174757282)^{-10} + \frac{31.84}{1474.18}(1.0194174757282)^{-11} + \frac{16.52}{1474.18}(1.0194174757282)^{-12} \\
&+ \frac{7.81}{1474.18}(1.0194174757282)^{-13} + \frac{3.30}{1474.18}(1.0194174757282)^{-14} + \frac{1.21}{1474.18}(1.0194174757282)^{-15} \\
&+ \frac{0.37}{1474.18}(1.0194174757282)^{-16} + \frac{0.09}{1474.18}(1.0194174757282)^{-17} + \frac{0.01}{1474.18}(1.0194174757282)^{-18} \\
&+ \frac{0.00}{1474.18}(1.0194174757282)^{-19} \\
= &\ 3.12628138309
\end{aligned}$$

The EPV of the payments is therefore $20000(1.03)^{-1} \times 3.12628138309 = \$60,704.49$.

7. *A woman aged 64 is receiving a monthly pension of \$3,000 at the start of each month. She wants to change this to an annual pension. If the appropriate life table is the ultimate part of Table 1 and the interest rate is $i = 0.04$, then we can calculate $A_{64} = 0.330027$. Use Woolhouse's formula to calculate the annual pension that has the same expected present value. [You may use the approximation $\mu_x = \frac{1}{2}(q_x + q_{x-1})$.]*

First we calculate $\ddot{a}_{64} = (1 - 0.330027)\frac{1.04}{0.04} = 17.419298$. Now from the lifetable we get

$$\begin{aligned}
\mu_{64} &= \frac{1}{2}(q_{63} + q_{64}) \\
&= \frac{1}{2}\left(\frac{33.44}{9638.51} + \frac{36.46}{9605.07}\right) \\
&= 0.00363266393331
\end{aligned}$$

Now Woolhouse's formula gives

$$\begin{aligned}
\ddot{a}_{64}^{(12)} &= \ddot{a}_{64} - \frac{11}{24} - \frac{143}{1728}(\log(1.04) + 0.00363266393331) \\
&= 17.419298 - \frac{11}{24} - \frac{143}{1728}(\log(1.04) + 0.00363266393331) \\
&= 16.9574183514
\end{aligned}$$

The EPV of the current benefit is therefore $16.9574183514 \times 3000 \times 12 = \$610,467.06$. To get an annual pension with the same benefit, the payments would be

$$\frac{16.9574183514 \times 3000 \times 12}{\ddot{a}_{64}} = \frac{16.9574183514 \times 3000 \times 12}{17.419298} = \$35,045.45$$

Table 1: Select lifetable to be used for questions on this assignment

| $x$ | $l_{[x]}$ | $l_{[x]+1}$ | $l_{[x]+2}$ | $l_{[x]+3}$ |
|---|---|---|---|---|
| 25 | 9998.75 | 9997.65 | 9996.30 | 9994.66 |
| 26 | 9997.00 | 9995.83 | 9994.40 | 9992.66 |
| 27 | 9995.14 | 9993.90 | 9992.38 | 9990.52 |
| 28 | 9993.16 | 9991.84 | 9990.22 | 9988.24 |
| 29 | 9991.05 | 9989.65 | 9987.92 | 9985.80 |
| 30 | 9988.81 | 9987.30 | 9985.46 | 9983.18 |
| 31 | 9986.40 | 9984.80 | 9982.82 | 9980.38 |
| 32 | 9983.83 | 9982.11 | 9979.99 | 9977.37 |
| 33 | 9981.07 | 9979.23 | 9976.95 | 9974.13 |
| 34 | 9978.11 | 9976.13 | 9973.68 | 9970.64 |
| 35 | 9974.93 | 9972.79 | 9970.16 | 9966.88 |
| 36 | 9971.50 | 9969.20 | 9966.36 | 9962.82 |
| 37 | 9967.80 | 9965.33 | 9962.25 | 9958.44 |
| 38 | 9963.81 | 9961.14 | 9957.82 | 9953.69 |
| 39 | 9959.50 | 9956.61 | 9953.02 | 9948.55 |
| 40 | 9954.84 | 9951.71 | 9947.82 | 9942.98 |
| 41 | 9949.79 | 9946.41 | 9942.19 | 9936.94 |
| 42 | 9944.32 | 9940.66 | 9936.08 | 9930.38 |
| 43 | 9938.39 | 9934.41 | 9929.45 | 9923.26 |
| 44 | 9931.96 | 9927.64 | 9922.25 | 9915.52 |
| 45 | 9924.97 | 9920.28 | 9914.42 | 9907.10 |
| 46 | 9917.37 | 9912.28 | 9905.91 | 9897.94 |
| 47 | 9909.11 | 9903.58 | 9896.65 | 9887.98 |
| 48 | 9900.13 | 9894.11 | 9886.57 | 9877.13 |
| 49 | 9890.36 | 9883.80 | 9875.59 | 9865.30 |
| 50 | 9879.71 | 9872.57 | 9863.63 | 9852.42 |
| 51 | 9868.12 | 9860.34 | 9850.59 | 9838.38 |
| 52 | 9855.48 | 9847.01 | 9836.39 | 9823.08 |
| 53 | 9841.72 | 9832.48 | 9820.90 | 9806.39 |
| 54 | 9826.71 | 9816.64 | 9804.02 | 9788.18 |
| 55 | 9810.34 | 9799.37 | 9785.60 | 9768.33 |
| 56 | 9792.49 | 9780.52 | 9765.51 | 9746.67 |
| 57 | 9773.03 | 9759.97 | 9743.60 | 9723.05 |
| 58 | 9751.79 | 9737.56 | 9719.69 | 9697.28 |
| 59 | 9728.63 | 9713.10 | 9693.62 | 9669.17 |
| 60 | 9703.36 | 9686.43 | 9665.17 | 9638.51 |
| 61 | 9675.80 | 9657.33 | 9634.15 | 9605.07 |
| 62 | 9645.73 | 9625.59 | 9600.31 | 9568.61 |
| 63 | 9612.94 | 9590.98 | 9563.42 | 9528.85 |
| 64 | 9577.18 | 9553.24 | 9523.19 | 9485.52 |
| 65 | 9538.19 | 9512.09 | 9479.35 | 9438.30 |
| 66 | 9495.69 | 9467.25 | 9431.58 | 9386.86 |
| 67 | 9449.37 | 9418.39 | 9379.54 | 9330.85 |
| 68 | 9398.90 | 9365.17 | 9322.87 | 9269.88 |
| 69 | 9343.95 | 9307.23 | 9261.20 | 9203.55 |
| 70 | 9284.12 | 9244.18 | 9194.11 | 9131.43 |
| 71 | 9219.03 | 9175.59 | 9121.17 | 9053.07 |
| 72 | 9148.24 | 9101.03 | 9041.91 | 8967.97 |
| 73 | 9071.30 | 9020.03 | 8955.85 | 8875.63 |
| 74 | 8987.73 | 8932.10 | 8862.49 | 8775.52 |
| 75 | 8897.04 | 8836.71 | 8761.27 | 8667.10 |
| 76 | 8798.69 | 8733.34 | 8651.66 | 8549.78 |
| 77 | 8692.13 | 8621.41 | 8533.09 | 8423.00 |
| 78 | 8576.81 | 8500.36 | 8404.95 | 8286.16 |
| 79 | 8452.13 | 8369.60 | 8266.68 | 8138.66 |
| 80 | 8317.52 | 8228.53 | 8117.67 | 7979.93 |
| 81 | 8172.36 | 8076.57 | 7957.35 | 7809.41 |
| 82 | 8016.08 | 7913.13 | 7785.15 | 7626.56 |
| 83 | 7848.11 | 7737.67 | 7600.54 | 7430.89 |
| 84 | 7667.89 | 7549.66 | 7403.05 | 7221.99 |
| 85 | 7474.92 | 7348.64 | 7192.27 | 6999.51 |
| 86 | 7268.77 | 7134.21 | 6967.86 | 6763.22 |
| 87 | 7049.07 | 6906.07 | 6729.62 | 6513.04 |
| 88 | 6815.55 | 6664.05 | 6477.46 | 6249.02 |
| 89 | 6568.09 | 6408.10 | 6211.48 | 5971.42 |
| 90 | 6306.70 | 6138.35 | 5931.96 | 5680.73 |
| 91 | 6031.59 | 5855.15 | 5639.41 | 5377.67 |
| 92 | 5743.19 | 5559.08 | 5334.61 | 5063.27 |
| 93 | 5442.15 | 5250.97 | 5018.61 | 4738.86 |
| 94 | 5129.44 | 4931.97 | 4692.79 | 4406.12 |
| 95 | 4806.33 | 4603.54 | 4358.89 | 4067.08 |
| 96 | 4474.39 | 4267.51 | 4018.96 | 3724.10 |
| 97 | 4135.60 | 3926.04 | 3675.44 | 3379.91 |
| 98 | 3792.25 | 3581.66 | 3331.11 | 3037.57 |
| 99 | 3447.02 | 3237.23 | 2989.05 | 2700.39 |
| 100 | 3102.90 | 2895.94 | 2652.63 | 2371.88 |
| 101 | 2763.19 | 2561.21 | 2325.37 | 2055.64 |
| 102 | 2431.39 | 2236.61 | 2010.90 | 1755.27 |
| 103 | 2111.15 | 1925.80 | 1712.81 | 1474.18 |
| 104 | 1806.12 | 1632.34 | 1434.48 | 1215.44 |
| 105 | 1519.82 | 1359.55 | 1178.94 | 981.65 |
| 106 | 1255.46 | 1110.36 | 948.70 | 774.71 |
| 107 | 1015.81 | 887.14 | 745.58 | 595.71 |
| 108 | 802.96 | 691.49 | 570.56 | 444.87 |
| 109 | 618.23 | 524.17 | 423.71 | 321.41 |
| 110 | 462.04 | 385.00 | 304.13 | 223.65 |
| 111 | 333.80 | 272.80 | 210.00 | 149.10 |
| 112 | 231.99 | 185.53 | 138.71 | 94.62 |
| 113 | 154.19 | 120.34 | 87.07 | 56.74 |
| 114 | 97.30 | 73.90 | 51.50 | 31.84 |
| 115 | 57.78 | 42.55 | 28.41 | 16.52 |
| 116 | 31.92 | 22.69 | 14.43 | 7.81 |
| 117 | 16.15 | 11.04 | 6.63 | 3.30 |
| 118 | 7.34 | 4.79 | 2.69 | 1.21 |
| 119 | 2.90 | 1.79 | 0.93 | 0.37 |
| 120 | 0.95 | 0.55 | 0.26 | 0.09 |
| 121 | 0.23 | 0.13 | 0.05 | 0.01 |
| 122 | 0.03 | 0.02 | 0.01 | 0.00 |