ACSC/STAT 3720, Life Contingencies I
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Homework Sheet 5
Model Solutions

Basic Questions

1. *An insurance company offers a whole life insurance policy with benefit \$500,000 payable at the end of the year of death. The premium for this policy for a select individual aged 39 for whom the lifetable in Table 1 is appropriate, is \$1447.23, payable at the start of each year. If the current interest rate is $i = 0.07$, what is the probability that the present value of future loss for this policy exceeds \$250,000?*

   If the policyholder dies in the nth year, then the PVFL is

   $$500000(1.07)^{-n} - 1447.23\frac{1.07}{0.07}(1 - 1.07^{-n}) = 522121.944286(1.07)^{-n} - 22121.9442857$$

   We solve for values of n such that

   $$\begin{aligned} 522121.944286(1.07)^{-n} - 22121.9442857 &\geqslant 250000 \\ 522121.944286(1.07)^{-n} &\geqslant 272121.9442857 \\ (1.07)^{-n} &\geqslant 0.521184652865 \\ n &\leqslant \frac{-\log(0.521184652865)}{\log(1.07)} \\ n &\leqslant 9.63144985355 \end{aligned}$$

   So the PVFL exceeds \$250,000 if the life dies within the first 9 years. The probability of this is $1 - \frac{9907.10}{9959.50} = 0.005261308299$.

2. *An insurance company offers a 5-year endowment insurance policy with death benefit \$300,000 payable at the end of the year of death. If the interest rate is $i = 0.04$, calculate the annual premium for this policy for a select individual aged 34, using the lifetable in Table 1 and the equivalence principle.*

   Using the standard recurrence, we calculate

   $$\begin{aligned} A_{39:\overline{0}|} &= 1 \\ A_{38:\overline{1}|} &= 0.961538 \\ A_{37:\overline{2}|} &= 0.92457 \\ A_{36:\overline{3}|} &= 0.889035 \\ A_{35:\overline{4}|} &= 0.854876 \\ A_{34:\overline{5}|} &= 0.822038 \end{aligned}$$

   This gives $\ddot{a}_{34:\overline{5}|} = \frac{1.04}{0.04}(1 - 0.822038) = 4.627012$. The premium for this policy is therefore $\frac{300000 \times 0.822038}{4.627012} = \$53,298.20$.

3. *The current interest rate is $i = 0.06$. An individual aged 49 to whom the ultimate part of the lifetable in Table 1 applies, wants to purchase a whole life insurance policy. Premiums are payable until age 80. The benefit of this policy should be \$1,300,000 at the end of the year of death. The initial costs to the insurance company are \$3,000 plus 22% of the first premium. Renewal costs are 2% of subsequent premiums. Calculate the Gross annual premiums for this policy. You calculate $A_{49} = 0.101917$ and $A_{80} = 0.400802$.*

   For a premium P, we have

$$\begin{aligned} 1300000A_{49} + 3000 + 0.2P + 0.02P\ddot{a}_{49:\overline{31}|} &= P\ddot{a}_{49:\overline{31}|} \\ 1300000A_{49} + 3000 + 0.2P &= 0.98P\ddot{a}_{49:\overline{31}|} \\ (0.98\ddot{a}_{49:\overline{31}|} - 0.2)P &= 1300000A_{49} + 3000 \end{aligned}$$

   We calculate,

$$\begin{aligned} \ddot{a}_{49:\overline{31}|} &= \ddot{a}_{49} - {}_{31}p_{49}(1.06)^{-31}\ddot{a}_{80} \\ &= \frac{1.06}{0.06}\left((1 - A_{49}) - \frac{8423.00}{9897.94}(1.06)^{-31}(1 - A_{80})\right) \\ &= \frac{1.06}{0.06}\left((1 - 0.101917) - \frac{8423.00}{9897.94}(1.06)^{-31}(1 - 0.400802)\right) \\ &= 14.3864621075 \end{aligned}$$

   This gives

$$\begin{aligned} (0.98 \times 14.3864621075 - 0.2)P &= 1300000 \times 0.101917 + 300013.8987328654P \quad = 135492.1 \\ P &= \$9,748.52 \end{aligned}$$

Standard Questions

4. *A select individual aged 37, to whom the lifetable in Table 1 applies, wants to purchase a whole life insurance policy with a death benefit of \$800,000. She can afford to pay a single lump-sum initial premium of \$40,000 and level annual premiums each subsequent year. The interest rate is $i = 0.04$, which gives $A_{[37]} = 0.1178276$. Use the equivalence principle to calculate net premiums for each year after the first.*

   The equivalence principle gives:

$$40000 + a_{[37]}P = 800000A_{[37]}$$

   We have $a_{[37]} = \ddot{a}_{[37]} - 1 = \frac{1.04}{0.04}(1 - 0.1178276) - 1 = 21.9364824$. Therefore the premium is the solution to

$$\begin{aligned} 40000 + 21.9364824P &= 800000 \times 0.1178276 \\ 21.9364824P &= 54262.08 \\ P &= \$2,473.60 \end{aligned}$$

5. *An individual aged 52 is paying premiums of \$200 a month for a 10-year endowment insurance policy which pays benefits at the end of the month of death. The individual's mortality follows the ultimate part of Table 1, and the interest rate is $i^{(12)} = 0.045$, so that $A_{52} = 0.18154$ and $A_{62} = 0.265995$. Calculate the equivalent annual premiums (for a policy which pays benefits at the end of the year of death, and has the same death benefit) using:*

   (a) Uniform distribution of deaths

   Under UDD, we get $A_{52}^{(12)} = \frac{i}{i^{(12)}} A_{52} = \frac{1.00375^{12}-1}{0.045} \times 0.18154 = 0.185331463062$ and $A_{62}^{(12)} = \frac{1.00375^{12}-1}{0.045} \times 0.265995 = 0.271550305812$. This gives

   $$\ddot{A}_{52:\overline{10}|}^{(12)} = A_{52}^{(12)} + {}_{10}p_{52}(1.00375)^{-120}(1 - A_{62}^{(12)}) = 0.185331463062 + \frac{9669.17}{9865.30}(1.00375)^{-120}(1 - 0.271550305812) = 0.$$

   We thus calculate

   $$\ddot{a}_{52:\overline{10}|}^{(12)} = \frac{1.00375}{0.045}(1 - 0.640960580482) = 8.00857371871$$

   The benefit of the policy is therefore

   $$200 \times 12 \times \frac{a_{52:\overline{10}|}^{(12)}}{A_{52:\overline{10}|}^{(12)}} = 200 \times 12 \times \frac{8.00857371871}{0.640960580482} = \$29,987.1435314$$

   For an annual policy with the same benefit, the premium is

   $$\frac{29987.1435314 \times A_{52:\overline{10}|}}{\ddot{a}_{52:\overline{10}|}}$$

   We calculate $A_{52:\overline{10}|} = A_{52} + {}_{10}p_{52}(1.00375)^{-120}(1 - A_{62}) = 0.18154 + \frac{9669.17}{9865.30}(1.00375)^{-120}(1 - 0.265995) = 0.640643837919$ and $d = \frac{(1.00375)^{12}-1}{(1.00375)^{12}} = 0.043922053583$, so $\ddot{a}_{52:\overline{10}|} = \frac{1-0.640643837919}{0.04593982504} = 7.8223232624$. This gives that the equivalent annual premium is

   $$\frac{29987.1435314 \times 0.640643837919}{7.8223232624} = \$2,455.93$$

   (b) Woolhouse's formula

   We calculate $\ddot{a}_{52} = \frac{1-0.18154}{0.043922053583} = 18.6343746076$ and $\ddot{a}_{62} = \frac{1-0.265995}{0.043922053583} = 16.7115364634$. We also get

   $$\mu_{52} \approx \frac{q_{51} + q_{52}}{2} = \frac{1}{2}\left(\frac{11.83}{9877.13} + \frac{12.88}{9865.30}\right) = 0.00125165129371$$
   $$\mu_{62} \approx \frac{q_{61} + q_{62}}{2} = \frac{1}{2}\left(\frac{28.11}{9697.28} + \frac{30.66}{9669.17}\right) = 0.00303482698669$$

   Now Woolhouse's formula gives

   $$\ddot{a}_{52}^{(12)} = 18.6343746076 - \frac{11}{24} - \frac{143}{1728}(12\log(1.00375) + 0.00125165129371) = 18.172220701$$
   $$\ddot{a}_{62}^{(12)} = 16.7115364634 - \frac{11}{24} - \frac{143}{1728}(12\log(1.00375) + 0.00303482698669) = 16.2492349908$$

This gives

$$\ddot{a}^{(12)}_{52:\overline{10}|} = 18.172220701 - \frac{9669.17}{9865.30}(1.00375)^{-120} \times 16.2492349908 = 8.0086848293$$

and therefore

$$A^{(12)}_{52:\overline{10}|} = 1 - d^{(12)}\ddot{a}^{(12)}_{52:\overline{10}|} = 1 - \frac{0.045}{1.00375} \times 8.0086848293 = 0.640955599185$$

The benefit of the policy is therefore

$$12 \times 200 \times \frac{8.0086848293}{0.640955599185} = 29987.7926251$$

The premium for an annual policy with this death benefit is therefore

$$\frac{29987.7926251 \times 0.640643837919}{7.8223232624} = \$2,455.98$$

6. *An insurance company provides a combined life insurance and annuity policy to an individual aged 46, using the ultimate part of the lifetable in Table 1. The interest rate is $i = 0.05$. This gives that $A_{46} = 0.123339$, and $A_{65} = 0.270592$. The individual will pay annual net premiums until age 65 (so the last premium will be at age 64). From age 65, they will receive an annuity of \$30,000 at the start of each year, and there is a \$250,000 death benefit at the end of the year of death. Premiums are calculated using the equivalence principle. What is the probability that the insurance company makes a net profit on this policy?*

   The EPV of the death benefits is $250000 \times 0.123339 = 30834.75$. The EPV of the annuity benefits is $30000 {}_{19}p_{46}(1.05)^{-19}\ddot{a}_{65} = 30000\frac{9568.61}{9923.26}(1.05)^{-19}\frac{1.05}{0.05}(1 - 0.270592) = 175351.252678$, so the total EPV of all benefits is 206186.002678. We also have $\ddot{a}_{46:\overline{19}|} = \ddot{a}_{46} - {}_{19}p_{46}(1.05)^{-19}\ddot{a}_{65} = \frac{1.05}{0.05}(1 - 0.123339) - \frac{9568.61}{9923.26}(1.05)^{-19}\frac{1.05}{0.05}(1 - 0.270592) = 12.5648392441$, so the premium is $\frac{206186.002678}{12.5648392441} = 16409.7604969$.

   If the life dies in the nth year, the present value of the death benefits is $250000(1.05)^{-n}$. If $n \leqslant 19$, then the present value of premiums paid is $16409.7604969\frac{1.05}{0.05}(1 - 1.05^{-n})$, so the profit is

$$16409.7604969\frac{1.05}{0.05}(1 - 1.05^{-n}) - 250000(1.05)^{-n} = 344604.970435 - 594604.970435(1.05)^{-n}$$

   This becomes positive when

$$\begin{aligned} 344604.970435 - 594604.970435(1.05)^{-n} &\geqslant 0 \\ 594604.970435(1.05)^{-n} &\leqslant 344604.970435 \\ (1.05)^n &\geqslant 1.72546835202 \\ n &\geqslant \frac{\log(1.72546835202)}{\log(1.05)} \\ n &\geqslant 11.1805018765 \end{aligned}$$

so if the life dies between the 12th and 19th year, then the policy is profitable. If the life survives to age 65, then the accumulated value of the premiums at that time is

$$16409.7604969\frac{1.05^{19}-1.05}{0.05} = 484727.979441$$

If the life dies in the nth year from age 65, then the PV at age 65 of the annuity and benefits is

$$30000\frac{1.05-1.05^{1-n}}{0.05}+250000(1.05)^{-n} = 630000-380000(1.05)^{-n}$$

. We therefore solve

$$\begin{aligned} 630000-380000(1.05)^{-n} &\leqslant 484727.979441 \\ (1.05)^{-n} &\geqslant \frac{630000-484727.979441}{380000} = 0.382294790945 \\ n &\leqslant \frac{-\log(0.382294790945)}{\log(1.05)} \\ n &\leqslant 19.7081374968 \end{aligned}$$

So the policy makes a profit if the life dies between ages 57 and 84. The probability of this is $\frac{9788.18-7809.41}{9923.26} = 0.199407251246$.

Table 1: Select lifetable to be used for questions on this assignment

x	$l_{[x]}$	$l_{[x]+1}$	$l_{[x]+2}$	$l_{[x]+3}$
25	9998.75	9997.65	9996.30	9994.66
26	9997.00	9995.83	9994.40	9992.66
27	9995.14	9993.90	9992.38	9990.52
28	9993.16	9991.84	9990.22	9988.24
29	9991.05	9989.65	9987.92	9985.80
30	9988.81	9987.30	9985.46	9983.18
31	9986.40	9984.80	9982.82	9980.38
32	9983.83	9982.11	9979.99	9977.37
33	9981.07	9979.23	9976.95	9974.13
34	9978.11	9976.13	9973.68	9970.64
35	9974.93	9972.79	9970.16	9966.88
36	9971.50	9969.20	9966.36	9962.82
37	9967.80	9965.33	9962.25	9958.44
38	9963.81	9961.14	9957.82	9953.69
39	9959.50	9956.61	9953.02	9948.55
40	9954.84	9951.71	9947.82	9942.98
41	9949.79	9946.41	9942.19	9936.94
42	9944.32	9940.66	9936.08	9930.38
43	9938.39	9934.41	9929.45	9923.26
44	9931.96	9927.64	9922.25	9915.52
45	9924.97	9920.28	9914.42	9907.10
46	9917.37	9912.28	9905.91	9897.94
47	9909.11	9903.58	9896.65	9887.98
48	9900.13	9894.11	9886.57	9877.13
49	9890.36	9883.80	9875.59	9865.30
50	9879.71	9872.57	9863.63	9852.42
51	9868.12	9860.34	9850.59	9838.38
52	9855.48	9847.01	9836.39	9823.08
53	9841.72	9832.48	9820.90	9806.39
54	9826.71	9816.64	9804.02	9788.18
55	9810.34	9799.37	9785.60	9768.33
56	9792.49	9780.52	9765.51	9746.67
57	9773.03	9759.97	9743.60	9723.05
58	9751.79	9737.56	9719.69	9697.28
59	9728.63	9713.10	9693.62	9669.17
60	9703.36	9686.43	9665.17	9638.51
61	9675.80	9657.33	9634.15	9605.07
62	9645.73	9625.59	9600.31	9568.61
63	9612.94	9590.98	9563.42	9528.85
64	9577.18	9553.24	9523.19	9485.52
65	9538.19	9512.09	9479.35	9438.30
66	9495.69	9467.25	9431.58	9386.86
67	9449.37	9418.39	9379.54	9330.85
68	9398.90	9365.17	9322.87	9269.88
69	9343.95	9307.23	9261.20	9203.55
70	9284.12	9244.18	9194.11	9131.43
71	9219.03	9175.59	9121.17	9053.07
72	9148.24	9101.03	9041.91	8967.97
73	9071.30	9020.03	8955.85	8875.63

x	$l_{[x]}$	$l_{[x]+1}$	$l_{[x]+2}$	$l_{[x]+3}$
74	8987.73	8932.10	8862.49	8775.52
75	8897.04	8836.71	8761.27	8667.10
76	8798.69	8733.34	8651.66	8549.78
77	8692.13	8621.41	8533.09	8423.00
78	8576.81	8500.36	8404.95	8286.16
79	8452.13	8369.60	8266.68	8138.66
80	8317.52	8228.53	8117.67	7979.93
81	8172.36	8076.57	7957.35	7809.41
82	8016.08	7913.13	7785.15	7626.56
83	7848.11	7737.67	7600.54	7430.89
84	7667.89	7549.66	7403.05	7221.99
85	7474.92	7348.64	7192.27	6999.51
86	7268.77	7134.21	6967.86	6763.22
87	7049.07	6906.07	6729.62	6513.04
88	6815.55	6664.05	6477.46	6249.02
89	6568.09	6408.10	6211.48	5971.42
90	6306.70	6138.35	5931.96	5680.73
91	6031.59	5855.15	5639.41	5377.67
92	5743.19	5559.08	5334.61	5063.27
93	5442.15	5250.97	5018.61	4738.86
94	5129.44	4931.97	4692.79	4406.12
95	4806.33	4603.54	4358.89	4067.08
96	4474.39	4267.51	4018.96	3724.10
97	4135.60	3926.04	3675.44	3379.91
98	3792.25	3581.66	3331.11	3037.57
99	3447.02	3237.23	2989.05	2700.39
100	3102.90	2895.94	2652.63	2371.88
101	2763.19	2561.21	2325.37	2055.64
102	2431.39	2236.61	2010.90	1755.27
103	2111.15	1925.80	1712.81	1474.18
104	1806.12	1632.34	1434.48	1215.44
105	1519.82	1359.55	1178.94	981.65
106	1255.46	1110.36	948.70	774.71
107	1015.81	887.14	745.58	595.71
108	802.96	691.49	570.56	444.87
109	618.23	524.17	423.71	321.41
110	462.04	385.00	304.13	223.65
111	333.80	272.80	210.00	149.10
112	231.99	185.53	138.71	94.62
113	154.19	120.34	87.07	56.74
114	97.30	73.90	51.50	31.84
115	57.78	42.55	28.41	16.52
116	31.92	22.69	14.43	7.81
117	16.15	11.04	6.63	3.30
118	7.34	4.79	2.69	1.21
119	2.90	1.79	0.93	0.37
120	0.95	0.55	0.26	0.09
121	0.23	0.13	0.05	0.01
122	0.03	0.02	0.01	0.00