ACSC/STAT 3720, Life Contingencies I
Winter 2018
Toby Kenney
Homework Sheet 6
Model Solutions

Basic Questions

1. *An insurer issues 3,000 whole life insurance policies to standard lives aged 41. The appropriate interest rate is $i = 0.05$. The company calculates $A_{41} = 0.0989705$ and ${}^2A_{41} = 0.0190814$. If the death benefit is $600,000, what annual premium should the company charge using the portfolio percentile method with a 90% probability of making a profit?*

   If the premium is P, then if the life dies after n years, the profit of each policy is $P\frac{1.05}{0.05}(1 - 1.05^{-n}) - 600000(1.05)^{-n} = 21P - (21P + 600000)(1.05)^{-n}$. The EPV of the profit of each policy is $21P-(21P+600000)A_{41} = 21P-0.0989705(21P+600000) = 18.9216195P-59382.3$. The variance of present value of the profit is

   $$(21P+600000)^2({}^2A_{41}-(A_{41})^2) = (0.0190814-0.0989705^2)(21P+600000)^2 = 0.00928624012975(21P+600000)^2$$

   For the portfolio of 3000 policies, the mean profit is $3000(18.9216195P-59382.3) = 56764.8585P-178146900$ and the variance is $3000\times0.00928624012975(21P+600000)^2 = 27.8587203893(21P+600000)^2$. Using the normal approximation, the 10th percentile of the profit is

   $$\begin{aligned}
   &56764.8585P - 178146900 + \Phi^{-1}(0.1)\sqrt{27.8587203893(21P + 600000)^2}\\
   =&56764.8585P - 178146900 - 1.281552\sqrt{27.8587203893}(21P + 600000)^2\\
   =&56764.8585P - 178146900 - 6.7642058387(21P + 600000)^2\\
   =&56764.8585P - 178146900 - 6.7642058387(441P^2 + 25200000P + 360000000000)\\
   =&-2,983.01477487P^2 + 170,514,751.994P + 2,434,935,955,030
   \end{aligned}$$

   Setting this equal to zero gives the premium:

   $$2,983.01477487P^2 - 170,514,751.994P - 2,434,935,955,030 = 0$$

   $$P = \frac{170514751.994 + \sqrt{170514751.994 + 4 \times 2983.01}}{2 \times 2983.01477487}$$

   $$P = \$57,151.33$$

2. *You are given the following values of A_x for various ages and interest rates:*

	i					
x	*0.03*	*0.0383523*	*0.04157806*	*0.042*	*0.042434*	*0.0453502*
39	*0.217774*	*0.148766*	*0.129115*	*0.126773*	*0.124417*	*0.109840*
44	*0.250364*	*0.177239*	*0.155883*	*0.153319*	*0.150733*	*0.134635*
49	*0.287221*	*0.210551*	*0.187599*	*0.184822*	*0.182017*	*0.164441*
54	*0.328587*	*0.249189*	*0.224839*	*0.221871*	*0.218869*	*0.199932*
59	*0.374567*	*0.293519*	*0.268073*	*0.264950*	*0.261784*	*0.241694*

Using the lifetable in Table 1, and interest rate $i = 0.03$, calculate the net annual premium for a 10-year term insurance policy with benefit \$250,000, sold to a life aged 39, if:

(a) The life works in a hazardous environment, and has mortality 0.012 higher than normal.

For this life $\ddot{a}_{39:\overline{10}}$ is calculated as for a standard life at force of interest $\delta + 0.012$, which coresponds to interest rate $e^{0.012}(1+i) - 1 = e^{0.012}(1.03) - 1 = 0.04243445754$. This gives $d = \frac{0.04243445754}{1.04243445754} = 0.0407070748986$. From the table, we see we get $A_{39} = 0.124417$ and $A_{49} = 0.182017$ so $\ddot{a}_{39} = \frac{1-0.124417}{0.0407070748986} = 21.5093568423$ and $\ddot{a}_{49} = \frac{1-0.182017}{0.0407070748986} = 20.0943693949$. Now we recalculate A_{39} and A_{49} using the actual interest rate $i = 0.03$, to get $A_{39} = 1 - \frac{0.03}{1.03} \times 21.5093568423 = 0.37351387838$ and $A_{49} = 1 - \frac{0.03}{1.03} \times 20.0943693949 = 0.414727105004$. We now get $A^{1}_{39:\overline{10}|} = 0.37351387838 - e^{-0.12}(1.03)^{-10}\frac{9897.94}{9962.82} \times 0.414727105004 = 0.101596247214$ and $\ddot{a}_{39:\overline{10}|} = 21.5093568423 - e^{-0.12}(1.03)^{-10}\frac{9897.94}{9962.82} \times 20.0943693949 = 8.3343961038$ The premium is therefore $\frac{250000 \times 0.101596247214}{8.3343961038} = \$3,047.50$

(b) The life is an impaired life, and is treated as a life five years older than its actual age.

From the table we have $A_{44} = 0.250364$ and $A_{54} = 0.328587$. This gives $\ddot{a}_{44} = \frac{1.03}{0.03}(1 - 0.250364) = 25.7375026666$ and $\ddot{a}_{54} = \frac{1.03}{0.03}(1 - 0.328587) = 23.0518463333$. We now calculate

$$A_{44:\overline{10}|} = 0.250364 - {}_{10}p_{44}(1.03)^{-10}0.328587 = 0.250364 - \frac{9838.38}{9936.94}(1.03)^{-10}0.328587 = 0.008289493275$$

and

$$\ddot{a}_{44:\overline{10}|} = 25.7375026666 - {}_{10}p_{44}(1.03)^{-10}23.0518463333 = 25.7375026666 - \frac{9838.38}{9936.94}(1.03)^{-10}23.0518463333 = 8.75$$

The premium is therefore

$$\frac{250000 \times 0.008289493275}{8.7548943156} = \$236.71$$

3. *An insurance company has a whole life insurance policy for an individual aged 47. The death benefit of this policy is \$350,000, and the interest rate is $i = 0.04$. Premiums are payable until age 80. The insurance company calculates $A_{47} = 0.184808$, and $A_{80} = 0.526062$. Therefore, the net annual premium for the policy is \$3,529.59. What is the policy value if the life survives to age 62? [Use the ultimate part of the lifetable in Table 1.]*

   (a) Using the same basis as the premium basis, which gives $A_{62} = 0.309353$.

   Using this basis, we have $\ddot{a}_{62} = \frac{1.04}{0.04}(1 - 0.309353) = 17.956822$, and $\ddot{a}_{80} = \frac{1.04}{0.04}(1 - 0.526062) = 12.322388$. This gives $\ddot{a}_{62:\overline{18}} = 17.956822 - \frac{8423.00}{9669.17}(1.04)^{-18} \times 12.322388 = 12.6580848438$. The policy value is therefore $350000 \times 0.309353 - 3529.59 \times 12.6580848438 = \$63,595.70$.

   (b) Using the reserve basis $i = 0.03$, which gives $A_{80} = 0.609835$, $A_{62} = 0.404336$, $A_{47} = 0.271948$, $\ddot{a}_{80} = 13.39566$, $\ddot{a}_{62} = 20.45113$ and $\ddot{a}_{47} = 24.99645$.

   Using the reserve basis, we recalculate the premium. Under this basis, we get

$$\ddot{a}_{47:\overline{33}|} = \ddot{a}_{47} - \frac{8423.00}{9915.52}(1.03)^{-33}\ddot{a}_{80} = 24.99645 - \frac{8423.00}{9915.52}(1.03)^{-33} \times 13.39566 = 20.7061564684$$

   so the premium is

$$\frac{350000 \times 0.271948}{20.7061564684} = \$4,596.78744074$$

We now calculate

$$\ddot{a}_{62:\overline{18}|} = \ddot{a}_{62} - \frac{8423.00}{9669.17}(1.03)^{-18}\ddot{a}_{80} = 20.45113 - \frac{8423.00}{9669.17}(1.03)^{-18} \times 13.39566 = 13.5966947838$$

so the policy value is

$$350000 \times 0.404336 - 4596.78744074 \times 13.5966947838 = \$79,016.48$$

4. *A life aged 47 takes out a whole life insurance with benefit \$400,000. The initial cost of this insurance is \$3,000 plus 30% of the first annual premium. The renewal cost is 3% of each subsequent premium. The interest rate is $i = 0.06$. Using the ultimate part of the lifetable in Table 1, we can calculate $A_{47} = 0.0921683$.*

   (a) Calculate the gross premium for this policy.

   We calculate $\ddot{a}_{47} = \frac{1.06}{0.06}(1 - 0.0921683) = 16.0383600334$. The gross premium P satisfies

$$\begin{aligned} 0.97 \times 16.0383600334P - 0.27P &= 400000 \times 0.0921683 + 3000 \\ 15.2872092324P &= 39867.32 \\ P &= \$2,607.89 \end{aligned}$$

   (b) Calculate the gross policy value after 2 years.

   We have ${}_0V = 0$,

$$\begin{aligned} 0 &= (1.06)^{-1}({}_1V + 400000q_{47}) - 2607.89 \\ {}_1V &= (1.06)2607.89 - 400000\frac{8.42}{9915.52} \\ &= 2424.69387183 \\ 2424.69387183 &= (1.06)^{-1}({}_2V + 400000q_{48}) - 2607.89 \\ {}_2V &= (1.06)(2424.69387183 + 2607.89) - 400000\frac{9.16}{9907.10} \\ &= \$4,964.70 \end{aligned}$$

Standard Questions

5. *A life insurance company sells 10-year term insurance policies to lives aged 53 for whom the Ultimate part of the lifetable in Table 1 is appropriate. The death benefit is \$400,000. The interest rate is $i = 0.06$. This gives $A_{53} = 0.124241$ and $A_{63} = 0.199371$, and also ${}^2A_{53} = 0.0311226$ and ${}^2A_{63} = 0.0655426$. Using the portfolio premium principle with a 95% probability of profit, they want to ensure the annual premium is under \$2,000. How many policies do they need to include in the portfolio to ensure this?*

   For each policy, if the policyholder dies in year n, for $n \leqslant 10$, then the profit is $P\frac{1.06}{0.06}(1 - 1.06^{-n}) - 400000(1.06)^{-n}$. If the policyholder survives to the end, the profit is $\frac{1.06}{0.06}(1 -$

$1.06^{-10})P = 7.80169227452P$. We have that the expected profit is therefore

$$\begin{aligned}
&{}_{10}p_{53}7.80169227452P +{}_{10} q_{53}\frac{1.06}{0.06}P - \left(\frac{1.06}{0.06}P + 400000\right) A^1_{53:\overline{10}|}\\
=&7.7014990931P + 0.226884049784P - (0.253393008922P + 5737.200202)\\
=&7.67499013396P - 5737.200202
\end{aligned}$$

Let Z be an indicator variable with $Z = 1$ if the life survives to the end of the 10 years, and $Z = 0$ otherwise. If the life dies in the Nth year, then the profit is

$$(1 - Z)\left(\frac{1.06}{0.06}P - \left(\frac{1.06}{0.06}P - 400000\right)(1.06)^{-N}\right) + 7.80169227452PZ$$

$$= (1 - Z)\left(\frac{1.06}{0.06}P - 7.80169227452P - \left(\frac{1.06}{0.06}P - 400000\right)(1.06)^{-n}\right) + 7.80169227452P$$

We have that $\mathrm{Var}(1 - Z) ={}_{10} p_{47 10}q_{47}$, $\mathrm{Var}((1.06)^{-N}(1 - Z)) ={}^2 A_{37:\overline{10}|} - (A_{37:\overline{10}|})^2$ and

$$\begin{aligned}
\mathrm{Cov}((1.06)^{-N}(1 - Z), (1 - Z)) &= \mathbb{E}((1.06)^{-N}(1 - Z)^2) - \mathbb{E}((1.06)^{-N}(1 - Z))\mathbb{E}(1 - Z)\\
&= \mathbb{E}((1.06)^{-N}(1 - Z)) - \mathbb{E}((1.06)^{-N}(1 - Z))_{10}q_{47}\\
&= \mathbb{E}((1.06)^{-N}(1 - Z))_{10}p_{47}\\
&={}_{10} p_{47}A^1_{47:\overline{10}|}
\end{aligned}$$

The variance of the profit on the policy is therefore

$$\begin{aligned}
&\left(\frac{1.06}{0.06} - 7.80169227452\right)^2 P^2{}_{10}p_{47 10}q_{47} + \left(\frac{1.06}{0.06}P - 400000\right)^2\left({}^2A_{37:\overline{10}|} - (A_{37:\overline{10}|})^2\right)\\
&\quad + 2\left(\frac{1.06}{0.06} - 7.80169227452\right)\left(\frac{1.06}{0.06}P - 400000\right){}_{10}p_{47}A^1_{47:\overline{10}|}
\end{aligned}$$

We calculate $A^1_{47:\overline{10}|} = 0.124241 - \frac{9788.18}{9915.52}(1.06)^{-10}0.199371 = 0.014343000505$ and ${}^2A^1_{47:\overline{10}|} = 0.0311226 - \frac{9788.18}{9915.52}(1.06)^{-20}0.0655426 = 0.0109485630272$. The variance of profit is therefore

$$\begin{aligned}
&1.23375159602P^2 + 0.0107428413637\left(\frac{1.06}{0.06}P - 400000\right)^2 + 0.279352411\left(\frac{1.06}{0.06}P - 400000\right)\\
=&4.58671175054P^2 - 14133328.3982P + 159999888259
\end{aligned}$$

When $P = 2000$, the expected profit is $7.67499013396 \times 2000 - 5737.200202 = 9612.7800659$ and the variance is $4.58671175054*2000^2-14133328.3982*2000+159999888259 = 131751578310$

Let the number of policies be m. We set the 5th percentile of profit to zero:

$$\begin{aligned}
9612.7800659m - 1.644854\sqrt{131751578310m} &= 0\\
1.644854\sqrt{131751578310m} &= 9612.7800659m\\
1.644854^2 \times 131751578310m &= 9612.7800659^2m^2\\
m &= \frac{1.644854^2 \times 131751578310}{9612.7800659^2}\\
&= 3857.55853659
\end{aligned}$$

so they need at least 3,858 policies.

6. *A life insurance company sells whole life insurance policies to lives aged 43. The interest rate is $i = 0.05$. Under the equivalence principle, the annual premium for a policy with benefit \$100,000 would be \$367.42. Using the portfolio premium percentile principle on a portfolio of 2,000 policies with benefit \$100,000, and a 95% probability of profit, they calculate a premium of \$394.03. What premium would they calculate if they used a portfolio of 1,600 policies with benefit \$100,000 and 200 policies with benefit \$200,000 (assume that premium is proportional to benefit, so these policies would pay double the premium of the others), and a 90% probability of profit.*

   For a whole-life insurance, the expected profit is $\ddot{a}_xP-100000A_x = \frac{1+i}{i}(1-A_x)P-100000A_x$, and the variance of profit is $\left(\frac{1+i}{i}P+100000\right)^2\left({}^2A_x-(A_x)^2\right)$. We are given that the when $P = 367.42$, the expected profit is zero, so $367.42\frac{1+i}{i}(1-A_x)-100000A_x = 0$, so $A_x = \frac{\left(367.42\frac{1+i}{i}\right)}{\left(367.42\frac{1+i}{i}+100000\right)} = \frac{367.42(1+i)}{367.42+100367.42i} = \frac{367.42(1.05)}{367.42+0.05\times100367.42} = 0.0716312608491$

   On the other hand, with a portfolio of 2,000 policies, using the portfolio percentile matching principle with a 95% probability of profit, the premium P is the solution to

$$\begin{aligned}
2000\left(\frac{1+i}{i}(1-A_x)P-100000A_x\right)-1.644854\sqrt{2000\left(\frac{1+i}{i}P+100000\right)^2\left({}^2A_x-(A_x)^2\right)} &= 0\\
2000\left(\frac{1+i}{i}(1-A_x)P-100000A_x\right)-1.644854\sqrt{2000\left({}^2A_x-(A_x)^2\right)}\left(\frac{1+i}{i}P+100000\right) &= 0
\end{aligned}$$

   We are given that this solution is $P = 394.03$, which gives

$$\begin{aligned}
2000\left(\frac{1+i}{i}(1-A_x)394.03-100000A_x\right) &= 1.644854\sqrt{2000\left({}^2A_x-(A_x)^2\right)}\left(\frac{1+i}{i}394.03+100000\right)\\
\frac{\left(\frac{1+i}{i}394.03+100000\right)}{\left(\frac{1+i}{i}(1-A_x)394.03-100000A_x\right)} &= \frac{2000}{1.644854\sqrt{2000\left({}^2A_x-(A_x)^2\right)}}
\end{aligned}$$

   Finally, when they assess a portfolio of of 1,600 policies with benefit \$100,000 and 200 policies with benefit \$200,000, the expected profit is $2000\left(\frac{1+i}{i}(1-A_x)P-100000A_x\right)$, but the variance of profit is

$$
\begin{aligned}
&1600\left(\frac{1+i}{i}P+100000\right)^2\left({}^2A_x-(A_x)^2\right)+200\times 4\left(\frac{1+i}{i}P+100000\right)^2\left({}^2A_x-(A_x)^2\right)\\
=&2400\left(\frac{1+i}{i}P+100000\right)^2\left({}^2A_x-(A_x)^2\right)
\end{aligned}
$$

The premium for this portfolio is therefore the solution to

$$
\begin{aligned}
2000\left(\frac{1+i}{i}(1-A_x)P-100000A_x\right)&=1.281552\sqrt{2400\left({}^2A_x-(A_x)^2\right)}\left(\frac{1+i}{i}P+100000\right)\\
\frac{\left(\frac{1+i}{i}P+100000\right)}{\left(\frac{1+i}{i}(1-A_x)P-100000A_x\right)}&=\frac{2000}{1.281552\sqrt{2400\left({}^2A_x-(A_x)^2\right)}}\\
&=\frac{1.644854\sqrt{2000}}{1.281552\sqrt{2400}}\frac{\left(\frac{1+i}{i}394.03+100000\right)}{\left(\frac{1+i}{i}(1-A_x)394.03-100000A_x\right)}\\
&=1.17165702159\frac{\left(\frac{1+i}{i}394.03+100000\right)}{\left(\frac{1+i}{i}(1-A_x)394.03-100000A_x\right)}\\
\frac{\left(\frac{1+i}{i}P+100000\right)}{\left(\frac{1+i}{i}(1-A_x)(P-367.42)\right)}&=1.17165702159\frac{\left(\frac{1+i}{i}394.03+100000\right)}{\left(\frac{1+i}{i}(1-A_x)394.03-100000A_x\right)}\\
\frac{(21P+100000)}{(21(1-0.0716312608491)(P-367.42))}&=1.17165702159\frac{(21\times 394.03+100000)}{(21(1-0.0716312608491)394.03-100000\times 0.0716312\ldots}\\
&=244.535846015\\
21P+100000&=4767.40813589(P-367.42)\\
4746.40813589P&=1851641.09729\\
P&=390.114175662
\end{aligned}
$$

Table 1: Select lifetable to be used for questions on this assignment

x	$l_{[x]}$	$l_{[x]+1}$	$l_{[x]+2}$	$l_{[x]+3}$
25	9998.75	9997.65	9996.30	9994.66
26	9997.00	9995.83	9994.40	9992.66
27	9995.14	9993.90	9992.38	9990.52
28	9993.16	9991.84	9990.22	9988.24
29	9991.05	9989.65	9987.92	9985.80
30	9988.81	9987.30	9985.46	9983.18
31	9986.40	9984.80	9982.82	9980.38
32	9983.83	9982.11	9979.99	9977.37
33	9981.07	9979.23	9976.95	9974.13
34	9978.11	9976.13	9973.68	9970.64
35	9974.93	9972.79	9970.16	9966.88
36	9971.50	9969.20	9966.36	9962.82
37	9967.80	9965.33	9962.25	9958.44
38	9963.81	9961.14	9957.82	9953.69
39	9959.50	9956.61	9953.02	9948.55
40	9954.84	9951.71	9947.82	9942.98
41	9949.79	9946.41	9942.19	9936.94
42	9944.32	9940.66	9936.08	9930.38
43	9938.39	9934.41	9929.45	9923.26
44	9931.96	9927.64	9922.25	9915.52
45	9924.97	9920.28	9914.42	9907.10
46	9917.37	9912.28	9905.91	9897.94
47	9909.11	9903.58	9896.65	9887.98
48	9900.13	9894.11	9886.57	9877.13
49	9890.36	9883.80	9875.59	9865.30
50	9879.71	9872.57	9863.63	9852.42
51	9868.12	9860.34	9850.59	9838.38
52	9855.48	9847.01	9836.39	9823.08
53	9841.72	9832.48	9820.90	9806.39
54	9826.71	9816.64	9804.02	9788.18
55	9810.34	9799.37	9785.60	9768.33
56	9792.49	9780.52	9765.51	9746.67
57	9773.03	9759.97	9743.60	9723.05
58	9751.79	9737.56	9719.69	9697.28
59	9728.63	9713.10	9693.62	9669.17
60	9703.36	9686.43	9665.17	9638.51
61	9675.80	9657.33	9634.15	9605.07
62	9645.73	9625.59	9600.31	9568.61
63	9612.94	9590.98	9563.42	9528.85
64	9577.18	9553.24	9523.19	9485.52
65	9538.19	9512.09	9479.35	9438.30
66	9495.69	9467.25	9431.58	9386.86
67	9449.37	9418.39	9379.54	9330.85
68	9398.90	9365.17	9322.87	9269.88
69	9343.95	9307.23	9261.20	9203.55
70	9284.12	9244.18	9194.11	9131.43
71	9219.03	9175.59	9121.17	9053.07
72	9148.24	9101.03	9041.91	8967.97
73	9071.30	9020.03	8955.85	8875.63
74	8987.73	8932.10	8862.49	8775.52
75	8897.04	8836.71	8761.27	8667.10
76	8798.69	8733.34	8651.66	8549.78
77	8692.13	8621.41	8533.09	8423.00
78	8576.81	8500.36	8404.95	8286.16
79	8452.13	8369.60	8266.68	8138.66
80	8317.52	8228.53	8117.67	7979.93
81	8172.36	8076.57	7957.35	7809.41
82	8016.08	7913.13	7785.15	7626.56
83	7848.11	7737.67	7600.54	7430.89
84	7667.89	7549.66	7403.05	7221.99
85	7474.92	7348.64	7192.27	6999.51
86	7268.77	7134.21	6967.86	6763.22
87	7049.07	6906.07	6729.62	6513.04
88	6815.55	6664.05	6477.46	6249.02
89	6568.09	6408.10	6211.48	5971.42
90	6306.70	6138.35	5931.96	5680.73
91	6031.59	5855.15	5639.41	5377.67
92	5743.19	5559.08	5334.61	5063.27
93	5442.15	5250.97	5018.61	4738.86
94	5129.44	4931.97	4692.79	4406.12
95	4806.33	4603.54	4358.89	4067.08
96	4474.39	4267.51	4018.96	3724.10
97	4135.60	3926.04	3675.44	3379.91
98	3792.25	3581.66	3331.11	3037.57
99	3447.02	3237.23	2989.05	2700.39
100	3102.90	2895.94	2652.63	2371.88
101	2763.19	2561.21	2325.37	2055.64
102	2431.39	2236.61	2010.90	1755.27
103	2111.15	1925.80	1712.81	1474.18
104	1806.12	1632.34	1434.48	1215.44
105	1519.82	1359.55	1178.94	981.65
106	1255.46	1110.36	948.70	774.71
107	1015.81	887.14	745.58	595.71
108	802.96	691.49	570.56	444.87
109	618.23	524.17	423.71	321.41
110	462.04	385.00	304.13	223.65
111	333.80	272.80	210.00	149.10
112	231.99	185.53	138.71	94.62
113	154.19	120.34	87.07	56.74
114	97.30	73.90	51.50	31.84
115	57.78	42.55	28.41	16.52
116	31.92	22.69	14.43	7.81
117	16.15	11.04	6.63	3.30
118	7.34	4.79	2.69	1.21
119	2.90	1.79	0.93	0.37
120	0.95	0.55	0.26	0.09
121	0.23	0.13	0.05	0.01
122	0.03	0.02	0.01	0.00