ACSC/STAT 4703, Actuarial Models II
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Toby Kenney

Homework Sheet 1

Model Solutions

Basic Questions

1. *Loss amounts follow an exponential distribution with $\theta = 12,000$. The distribution of the number of losses is given in the following table:*

Number of Losses	*Probability*
0	*0.02*
1	*0.24*
2	*0.36*
3	*0.28*

Assume all losses are independent and independent of the number of losses. The insurance company buys excess-of-loss reinsurance on the part of the loss above $150,000. Calculate the expected payment for this excess-of-loss reinsurance.

If there are a total of n losses, the aggregate loss follows a gamma distribution with $\alpha = n$ and $\theta = 12000$. The expected payment on the excess-of-loss reinsurance is therefore

$$12000\left(\frac{\int_{12.5}^{\infty}(x-12.5)x^{n-1}e^{-x}}{\Gamma(n)}\right)$$

Integrating by parts repeatedly, we find that this is equal to

$$12000e^{-12.5}\left((n-12.5)\left(1+12.5+\frac{12.5^2}{2!}+\cdots+\frac{12.5^{n-1}}{(n-1)!}\right)+\frac{12.5^n}{(n-1)!}\right)$$

In particular

Number of Losses	Probability	expected payment	probability times expected payment
0	0.02	0	0
1	0.24	$12000e^{-12.5}$	$2880e^{-12.5}$
2	0.36	$174000e^{-12.5}$	$62640e^{-12.5}$
3	0.28	$1263500e^{-12.5}$	$356580e^{-12.5}$

So the total expected payment is $422100e^{-12.5} = 1.57302$.

2. *Aggregate payments have a compund distribution. The frequency distribution is negative binomial with $r = 3$ and $\beta = 6$. The severity distribution is a Pareto distribution with $\alpha = 6$ and $\theta = 12000$. Use a Gamma approximation to aggregate payments to estimate the probability that aggregate payments are more than \$100,000.*

   The mean of aggregate payments is $18 \times 2400 = \$43200$, and the variance is

   $$126 \times 2400^2 + 18 \times 7200000 = 881280000$$

   To get a Gamma distribution with the same mean and variance, we need $\theta = \frac{881280000}{43200} = 20400$ and $\alpha = \frac{43200}{20400} = \frac{36}{17}$. The probability that aggregate payments exceed \$1,000,000 is therefore the probability that a gamma distribution with $\alpha = \frac{36}{17}$ and $\theta = 20400$ exceeds 100,000, which is 0.05129455.

3. *An insurance company models loss frequency as negative binomial with $r = 4$, $\beta = 3$, and loss severity as exponential with $\theta = \$4,500$. Calculate the expected aggregate payments if there is a policy limit of \$50,000 and a deductible of \$1,000 applied to each claim.*

   With the policy limit and deductible, the expected payment per loss is

   $$\int_{1000}^{50000} e^{-\frac{x}{4500}}\,dx = 4500\int_{\frac{2}{9}}^{\frac{100}{9}} e^{-u}\,du = 4500(e^{-\frac{2}{9}} - e^{-\frac{100}{9}}) = 3603.251$$

   The expected number of losses is $4 \times 3 = 12$, so the expected aggregate payment is $12 \times 3603.251 = \$43,239.01$.

4. *Claim frequency follows a negative binomial distribution with $r = 8$ and $\beta = 1.7$. Claim severity (in thousands) has the following distribution:*

Severity	*Probability*
1	*0.5*
2	*0.3*
3	*0.15*
4	*0.03*
5	*0.015*
6	*0.004*
7	*0.0007*

   Use the recursive method to calculate the exact probability that aggregate claims are at least 8.

   The probability that aggregate claims are 0 is $\left(\frac{1}{2.7}\right)^8 = 0.0003540706$. For the negative binomial distribution we have $a = \frac{\beta}{1+\beta}$, $b = \frac{(r-1)\beta}{1+\beta}$. The recurrence is

$$
\begin{aligned}
f_S(x) &= \frac{(p_1 - (a+b)p_0)f_X(x) + \sum_{y=1}^{x\wedge m}\left(a + \frac{by}{x}\right) f_X(y) f_S(x-y)}{1 - af_X(0)} \\
&= \sum_{y=1}^{x\wedge m}\left(\frac{\beta}{1+\beta} + \frac{(r-1)\beta y}{(1+\beta)x}\right) f_X(y) f_S(x-y) \\
&= \frac{1.7}{2.7}\sum_{y=1}^{x}\left(1 + \frac{7y}{x}\right) f_X(y) f_S(x-y)
\end{aligned}
$$

This gives:

$$
\begin{aligned}
f_S(0) &= 0.0003540706 \\
f_S(1) &= \frac{1.7}{2.7} \times 8 \times 0.5 \times 0.0003540706 = 0.0008917334 \\
f_S(2) &= 0.0017983289 \\
f_S(3) &= 0.0031091403 \\
f_S(4) &= 0.0048001634 \\
f_S(5) &= 0.0068795627 \\
f_S(6) &= 0.0092934225 \\
f_S(7) &= 0.0119757438
\end{aligned}
$$

R code:

```
fs<-rep(0,8)
fs[1]<-0.0003540706
fx<-c(0.5,0.3,0.15,0.03,0.015,0.004,0.0007)
mat<-1+7*(1:7)%*%t(rep(1,7))/rep(1,7)%*%t(1:7)
newmat<-mat*fx%*%t(rep(1,7))
for(i in 1:7){
fs[i+1]<-t(fs[i:1])%*%(upper.tri(newmat,diag=TRUE)*newmat)[1:i,i]*1.7/2.7
}
sum(fs)
```

The total of these is 0.2598905 0.03910217, so the probability that aggregate claims are at least 8 is $1 - 0.03910217 = 0.96089783$.

5. *Using an arithmetic distribution ($h = 1$) to approximate a Pareto distribution with $\alpha = 4$ and $\theta = 9$, calculate the probability that the value is between 2.5 and 6.5, for the approximation using:*

   (a) The method of rounding.

The method of rounding sets the probability $P(A = n)$ to $P(n - 0.5 \leqslant X < n + 0.5)$, so in this case,

$$P(A \in \{3,4,5,6\}) = P(2.5 < X < 6.5) = \left(\frac{9}{9+2.5}\right)^4 - \left(\frac{9}{9+6.5}\right)^4 = 0.261458$$

(b) The method of local moment matching, matching 1 moment on each interval.

On the interval, $\left[2n - \frac{1}{2}, 2n + \frac{3}{2}\right]$, for $n \geqslant 1$, the probability of the Pareto distribution is

$$\frac{9^4}{\left(9 + 2n - \frac{1}{2}\right)^4} - \frac{9^4}{\left(9 + 2n + \frac{3}{2}\right)^4}$$

while the conditional mean is

$$\begin{aligned} & 2n - \frac{1}{2} + \frac{\left(\int_{-\frac{1}{2}}^{\frac{3}{2}} \frac{9^4}{(9+2n+x)^4}\,dx - \frac{9^4}{\left(9+2n+\frac{3}{2}\right)^4}\right)}{\left(\frac{9^4}{\left(9+2n-\frac{1}{2}\right)^4} - \frac{9^4}{\left(9+2n+\frac{3}{2}\right)^4}\right)} \\ &= 2n - \frac{1}{2} + \frac{\frac{1}{3}\left(\frac{1}{\left(9+2n-\frac{1}{2}\right)^3} - \frac{1}{\left(9+2n+\frac{3}{2}\right)^3}\right) - \frac{1}{\left(9+2n+\frac{3}{2}\right)^4}}{\left(\frac{1}{\left(9+2n-\frac{1}{2}\right)^4} - \frac{1}{\left(9+2n+\frac{3}{2}\right)^4}\right)} \end{aligned}$$

If the probability is ϕ and the conditional mean is μ, we match this by setting

$$\begin{aligned} p_{2n} + p_{2n+1} &= \phi \\ 2np_{2n} + (2n+1)p_{2n+1} &= \phi\mu \\ p_{2n+1} &= \phi(\mu - 2n) \\ &= 9^4\left(\frac{1}{3}\left(\frac{1}{\left(9+2n-\frac{1}{2}\right)^3} - \frac{1}{\left(9+2n+\frac{3}{2}\right)^3}\right) - \frac{2}{\left(9+2n+\frac{3}{2}\right)^4} - \frac{1}{2}\left(\frac{1}{\left(9+2n-\frac{1}{2}\right)\right.\right. \end{aligned}$$

In particular, we get

$$
\begin{aligned}
p_3 &= 0.09647344 \\
p_4 &= 0.1202779 \\
p_5 &= 0.00003870591 \\
p_6 &= 0.05988173
\end{aligned}
$$

So the total probability is $0.0000763192 + 0.1202779 + 0.00003870591 + 0.05988173 = 0.1802746$.

Standard Questions

6. *The number of claims an insurance company receives follows a negative binomial distribution with $r = 46$ and $\beta = 8.4$. Claim severity follows a negative binomial distribution with $r = 14$ and $\beta = 0.8$. Calculate the probability that aggregate losses exceed \$4000.*

   (a) Starting the recurrence 6 standard deviations below the mean.

   The mean is $46 \times 8.4 \times 14 \times 0.8 = 4327.68$. The variance is $46 \times 8.4 \times 14 \times 0.8 \times 1.8 + 46 \times 8.4 \times 9.4 \times (14 \times 0.8)^2 = 463407.9744$, so the standard deviation is $\sqrt{463407.9744} = 680.7408$. Six standard deviations below the mean is therefore $4327.68 - 6 \times 680.7408 = 243.2352$. We will therefore start the recurrence at $x = 243$. We set $f(243) = 1$ and $f(242) = 0$. We then apply the recurrence

$$
\begin{aligned}
f_S(x) &= \frac{(p_1 - (a+b)p_0) f_X(x) + \sum_{y=1}^{x \wedge m} \left(a + \frac{by}{x}\right) f_X(y) f_S(x-y)}{1 - a f_X(0)} \\
&= \frac{\left(\frac{8.4}{9.4}\right) \frac{1}{1.8^{14}} \sum_{y=1}^{x-243} \left(1 + \frac{45y}{x}\right) \binom{y+13}{y} \left(\frac{0.8}{1.8}\right)^y f_S(x-y)}{1 - \frac{8.4}{9.4} \left(\frac{1}{1.8}\right)^{14}} \\
&= 0.0002384734 \sum_{y=1}^{x-243} \left(1 + \frac{45y}{x}\right) \binom{y+13}{y} \left(\frac{0.8}{1.8}\right)^y f_S(x-y)
\end{aligned}
$$

   R code:

```
fs<-rep(0,10001)
fs[1]=1
y<-1:10000
fx<-(y+1)*(y+2)*(y+3)*(y+4)*(y+5)*(y+6)*(y+7)*(y+8)*(y+9)*(y+10)*(y+11)*(y+
x<-(1:10000)+243
mat<-1+45*(y)%*%t(rep(1,10000))/rep(1,10000)%*%t(x)
```

```
newmat<-mat*fx%*%t(rep(1,10000))
utnewmat<-upper.tri(newmat,diag=TRUE)*newmat
for(i in 1:10000){
fs[1+i]<-(8.4/9.4/1.8^14)/(1-(8.4/9.4/1.8^14))*t(fs[i:1])%*%utnewmat[1:i,i]
}
a<-sum(fs)
fs<-fs/a
sum(fs[3759:10001])
```

This gives the probability of the aggregate loss exceeding 4000 as 0.6710699.

(b) Using a suitable convolution.

If we convolve n times, we the distributions to be convolved are compound negative binomial with $r = \frac{46}{n}$, $\beta = 8.4$ and $r = 14$, $\beta = 0.8$. The pgf. of a negative binomial distribution is

$$P(z) = (1 - \beta(z - 1))^{-r}$$

The probability that the compound distribution is zero is therefore

$$\left(1 - 8.4(1.8^{-14} - 1)\right)^{-r} = 9.397759^{-r}$$

We want to choose r so that this is large enough to avoid underflow. For convenience, we will choose to convolve 4 times, so that $r = 11.5$. For $r = 11.5$ in the primary distribution, we compute the compound distribution from 0 to 4000 using the standard recurrence.

R code:

```
fs<-rep(0,4001)
fs[1]=(9.4-8.4/1.8^14)^(-10.5)
y<-1:4000
fx<-(y+1)*(y+2)*(y+3)*(y+4)*(y+5)*(y+6)*(y+7)*(y+8)*(y+9)*(y+10)*(y+11)*(y+
x<-(1:4000)
mat<-1+(10.5*(y)%*%t(rep(1,4000)))/(rep(1,4000)%*%t(x))
newmat<-mat*(fx%*%t(rep(1,4000)))
utnewmat<-upper.tri(newmat,diag=TRUE)*newmat
for(i in 1:4000){
fs[1+i]<-(8.4/9.4/1.8^14)/(1-(8.4/9.4/1.8^14))*t(fs[i:1])%*%utnewmat[1:i,i]
}
```

We then find the probabilities for values from 0 to 4000 by convolving this distribution with itself, then convolving the resulting distribution with itself.

R code:

```
ConvolveSelf<-function(n){
  convolution<-vector("numeric",2*length(n))
  for(i in 1:(length(n))){
```

```
    convolution[i]<-sum(n[1:i]*n[i:1])
  }
  for(i in 1:(length(n))){
    convolution[2*length(n)+1-i]<-sum(n[length(n)+1-(1:i)]*n[length(n)+1-(i:
  }
  return(convolution)
}

fs23<-ConvolveSelf(fs)
fs46<-ConvolveSelf(fs23)
sum(fs46[1:4000])
```

This gives the probability that aggregate losses are at most 4000 as 0.3283725, so the probability that aggregate losses exceed 4000 is 0.6716275.