ACSC/STAT 4703, Actuarial Models II
Fall 2015
Toby Kenney
Homework Sheet 6
Model Solutions

Basic Questions

1. *An insurance company sells car insurance. It estimates that the standard deviation of the aggregate annual claim for an individual is \$1,326 and the mean is \$1,102.*

   (a) How many years history are needed for an individual or group to be assigned full credibility? (Use $r = 0.05$, $p = 0.95$.)

   We want to determine the number N, so that after N years, the 95% confidence interval for this individual's average aggregate annual claim has width $2 \times 0.05 \times 1102$. That is, we need to solve

$$1.96\frac{1326}{\sqrt{N}} = 0.05 \times 1102$$
$$N = \left(\frac{1326 \times 1.96}{55.10}\right)^2 = 2224.826$$

   The standard premium for this policy is \$1,102. An individual has no claims in the last 10 years.

   (b) What is the Credibility premium for this individual, using limited fluctuation credibility?

   We have that the individual's credibility is $Z = \sqrt{\frac{10}{2224.826}} = 0.06704278$, so the credibility premium is $0 \times 0.06704278 + 1102 \times (1 - 0.06704278) =$ \$1,028.12.

2. *A health insurance company classifies individuals as healthy or unhealthy. Annual claims from healthy individuals follow a Gamma distribution with shape $\alpha = 0.25$ and scale $\theta = 1044$. Annual claims from unhealthy individuals follow a Gamma distribution with shape $\alpha = 0.5$ and scale $\theta = 1370$. 80% of individuals are healthy individuals.*

   (a) Calculate the expectation and variance of the aggregate annual claims from a randomly chosen individual.

   The expected aggregate annual claims from a healthy individual is $0.25 \times 1044 = 261$, and the variance is $0.25 \times 1044^2 = 272484$. The expected aggregate annual claims from an unhealthy individual is $0.5 \times 1370 = 685$, and the variance is $0.5 \times 1370^2 = 938450$.

The expected aggregate annual claims from a random individual is therefore $0.8 \times 261 + 0.2 \times 685 = \$345.80$. The variance is $0.8 \times 272484 + 0.2 \times 938450 + 0.8 \times 0.2 \times (685 - 261)^2 = 434441.36$.

(b) Given that an individual's total claims over the past 2 years are \$396, what are the expectation and variance of the individual's total claims next year?

For a healthy individual, the aggregate claims over 2 years follows a Gamma distribution with $\alpha = 0.5$ and $\theta = 1044$, which means the likelihood of aggregate claims being 396 is $\frac{396^{-0.5}1044^{-0.5}e^{-\frac{396}{1044}}}{\Gamma(0.5)} = 0.0006004749$.

For an unhealthy individual, the aggregate claims over 2 years follows a Gamma distribution with $\alpha = 1$ and $\theta = 1370$, which means the likelihood of aggregate claims being 396 is $\frac{1370^{-1}e^{-\frac{396}{1370}}}{\Gamma(1)} = 0.0005466963$.

The posterior probability that the individual is healthy is therefore $\frac{0.8\times 0.0006004749}{0.8\times 0.0006004749+0.2\times 0.0005466963} = 0.814591$.

The expected aggregate claims next year are therefore $0.814591 \times 261 + 0.185409 \times 685 = \$339.61$. The expected variance is $0.814591 \times 272484 + 0.185409 \times 938450 + 0.814591 \times 0.185409 \times (685 - 261)^2 = 423112.11$.

3. *The number of claims made by an individual in a year follows a Poisson distribution with mean Λ, where the value of Λ follows a Gamma distribution with $\alpha = 2.3$ and $\theta = 0.07$. Given that an individual has made 6 claims in the past 2 years, what is the expected number of claims made in the next year?*

   The probability of making 6 claims in 2 years is proportional to $e^{-2\lambda}\lambda^6$, so the posterior density of λ is proportional to $\lambda^{1.3}e^{-\frac{\lambda}{0.07}}\lambda^6 e^{-2\lambda} = \lambda^{7.3}e^{-\frac{\lambda}{\left(\frac{1}{2+\frac{1}{0.07}}\right)}}$, so the posterior distribution is a gamma distribution with $\alpha = 8.3$ and $\theta = \frac{1}{2+\frac{1}{0.07}} = \frac{0.07}{1.14} = 0.06140351$. The expected number of claims next year is the posterior mean of λ, which is $8.3 \times 0.06140351 = 0.5096491$.

Standard Questions

4. *For a certain insurance policy, the book premium is based on average claim frequency of 0.5 claims per year, and average claim severity of \$3,040. A particular group has made 60 claims from 187 policies in the last year. The average claim severity is \$3,914. Estimate the credibility premium for this group using limited fluctuation credibility if the standard for full credibility is:*

   (a) 203 claims for claim frequency, 700 claims for severity.

The credibility estimate for claim frequency uses $Z = \sqrt{\frac{60}{203}} = 0.5436603$, so the average claim frequency is $(1-0.5436603)\times 0.5+0.5436603\times\frac{60}{187} = 0.4026063$. The credibility estimate for claim severity uses $Z = \sqrt{\frac{60}{700}} = 0.29277$, so the credibility estimate for claim severity is $0.29277 \times 3914 + (1-0.29277)\times 3040 = \$3,295.88$. The credibility premium is therefore $0.4026063 \times 3295.88 = \$1,326.94$.

(b) 406 years for claim frequency, 700 claims for severity.

As in part (a), the credibility estimate for average severity is 3295.88. However, the credibility estimate for frequency now uses $Z = \sqrt{\frac{187}{406}} = 0.6786686$, so the average claim frequency is $0.6786686\times\frac{60}{187}+(1-0.6786686)\times 0.5 = 0.3784203$, so the credibility premium is $0.3784203 \times 3295.88 = \$1,247.23$.

(c) 523 years for aggregate claims.

The average aggregate claims is $\frac{60\times 3914}{187} = 1255.829$. The book value for expected aggregate claims is $0.5 \times 3040 = 1520$. The credibility is $Z = \sqrt{\frac{187}{523}} = 0.597957$, so the credibility premium is $0.597957\times 1255.829 + (1-0.597957)\times 1520 = \$1,362.04$.

5. *A group insurance policy covers 168 individuals. The insurance company reviews the last 3 years of aggregate claims for each insured. For individual i, the aggregate claims in year j are denoted X_{ij}. We have the following:*

$$\begin{aligned} \mathbb{E}(X_{ij}) &= \mu \\ \mathrm{Var}(X_{ij}) &= \sigma^2 \\ \mathrm{Cov}(X_{ij}, X_{kl}) &= \begin{cases} \rho & \text{if } i = k, j \neq l \\ \tau & \text{if } i \neq k, j = l \\ \zeta & \text{if } i \neq k, j \neq l \end{cases} \end{aligned}$$

Calculate the credibility estimate for $X_{i,4}$.

Let the estimate of $X_{i,4}$ be $\hat{X} = Z_0\mu + \sum_{j,k} Z_{jk}X_{jk}$. We will calculate the Z_{jk} to minimise the mean squared error of $\hat{X}$. That is, we want to minimise $\mathbb{E}((\hat{X}-X_{i,4})^2) = \mathbb{E}(\hat{X}^2+{X_{i,4}}^2-2\hat{X}X_{i,4})$. By subtracting μ from all terms, this becomes $\mathbb{E}((\hat{X}-\mu)^2+(X_{i,4}-\mu)^2-2(\hat{X}-\mu)(X_{i,4}-\mu))$, which is $\mathrm{Var}(\hat{X}) + \mathrm{Var}(X_{i,4}) - 2\,\mathrm{Cov}(\hat{X}, X_{i,4})$. Expanding these terms, we know that $\mathrm{Var}(X_{i,4}) = \sigma^2$, while $\mathrm{Cov}(\hat{X}, X_{i,4}) = \sum_{j,k} Z_{jk}\,\mathrm{Cov}(X_{jk}, X_{i,4}) = \sum_k Z_{ik}\rho + \sum_{j\neq i,k} Z_{jk}\zeta$. Finally

$$
\begin{aligned}
\mathrm{Var}(\hat{X}) &= \sum_{j,k,l,m} Z_{jk} Z_{lm} \,\mathrm{Cov}(X_{jk}, X_{lm}) \\
&= \sum_{j,k} {Z_{jk}}^2 \sigma^2 + \sum_{j,k,m} Z_{jk} Z_{jm} \rho + \sum_{j,k,l} Z_{j,k} Z_{l,k} \tau + \sum_{j \neq l, k \neq m} Z_{j,k} Z_{l,m} \zeta
\end{aligned}
$$

We therefore need to choose $Z_{j,k}$ to minimise

$$
\sum_{j,k} {Z_{jk}}^2 \sigma^2 + \sum_{j,k \neq m} Z_{jk} Z_{jm} \rho + \sum_{j \neq l,k} Z_{j,k} Z_{l,k} \tau + \sum_{j \neq l, k \neq m} Z_{j,k} Z_{l,m} \zeta \\
- \left(\sum_{k} Z_{ik} \rho + \sum_{j \neq i,k} Z_{jk} \zeta \right)
$$

Differentiating with respect to Z_{jk} gives in the cases $j = i$ and $j \neq i$ respectively, gives:

$$
\begin{aligned}
2Z_{ik}\sigma^2 + \sum_{m \neq k} Z_{im}\rho + \sum_{l \neq i} Z_{lk}\tau + \sum_{l \neq i, m \neq k} Z_{lm}\zeta = \rho \\
2Z_{jk}\sigma^2 + \sum_{m \neq k} Z_{jm}\rho + \sum_{l \neq j} Z_{lk}\tau + \sum_{l \neq i, m \neq k} Z_{lm}\zeta = \zeta
\end{aligned}
$$

By symmetry, we expect Z_{ik} to be the same for all $k = 1, 2, 3$. We can confirm this from the first equation. We also expect Z_{jk} to be the same for all $j \neq i$. Letting $Z_{ik} = a$ and $Z_{jk} = b$, our equations become:

$$
\begin{aligned}
(2\sigma^2 + 2\rho)a + 167b\tau + 167 \times 2b\zeta &= \rho \\
2b\sigma^2 + 2b\rho + 166b\tau + a\tau + 166 \times 2b\zeta + 2a\zeta &= \zeta \\
(2\sigma^2 + 2\rho)a + 167(\tau + 2\zeta)b &= \rho \\
(2\sigma^2 + 2\rho + 166\tau + 332\zeta)b + (\tau + 2\zeta)a &= \zeta \\
167(\tau + 2\zeta)^2 b - (2\sigma^2 + 2\rho)(2\sigma^2 + 2\rho + 166\tau + 332\zeta)b &= \rho(\tau + 2\zeta) - \zeta(2\sigma^2 + 2\rho) \\
b &= \frac{\rho(\tau + 2\zeta) - \zeta(2\sigma^2 + 2\rho)}{(\tau + 2\zeta)^2 - (2\sigma^2 + 2\rho)(2\sigma^2 + 2\rho + 166\tau + 332\zeta)} \\
((2\sigma^2 + 2\rho) - (\tau + 2\zeta))(a - b) &= \rho - \zeta \\
a &= b + \frac{\rho - \zeta}{((2\sigma^2 + 2\rho) - (\tau + 2\zeta))}
\end{aligned}
$$