MATH/STAT 4703, Actuarial Models II

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Formula Sheet

## General Mathematics

- Quadratic Formula: Solution to $ax^2+bx+c=0$ is $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$
- Gamma function: $\Gamma(\alpha)=\int_0^\infty x^\alpha e^{-x}dx$ satisfies $\Gamma(\alpha+1)=\alpha\Gamma(\alpha)$.

## Moments

Centralised moments in terms of uncentralised moments:

$$\begin{aligned}\mu_2&=\mu_2'-\mu^2\\ \mu_3&=\mu_3'-3\mu\mu_2'+2\mu^3\\ \mu_4&=\mu_4'-4\mu\mu_3'+6\mu^2\mu_2'-3\mu^4\end{aligned}$$

## Risk Measures

- Standard deviation principle $r=\mu+a\sigma$.
- Value at Risk $r=\pi_p$.
- Tail Value at Risk $r=\dfrac{\int_{\pi_p}^\infty xf(x)dx}{1-p}=\pi_p+\dfrac{\int_{\pi_p}^\infty S(x)dx}{1-p}$

## Continuous Distributions: Transformed Beta family

### Transformed Beta

Inverse of Transformed Beta with $\alpha=\tau$, $\tau=\alpha$, $\theta=\frac{1}{\theta}$.

| | |
|---|---|
| Density function | $f(x)=\left(\frac{\Gamma(\alpha+\tau)}{\Gamma(\alpha)\Gamma(\tau)}\right)\frac{\gamma\left(\frac{x}{\theta}\right)^{\tau\gamma}}{x\left(1+\left(\frac{x}{\theta}\right)^\gamma\right)^{\alpha+\tau}}$ |
| Mean | $\theta\frac{\Gamma\left(\tau+\frac{1}{\gamma}\right)\Gamma\left(\alpha-\frac{1}{\gamma}\right)}{\Gamma(\tau)\Gamma(\alpha)}$ |
| Raw Moments | $\mu_k'=\theta^k\frac{\Gamma\left(\tau+\frac{k}{\gamma}\right)\Gamma\left(\alpha-\frac{k}{\gamma}\right)}{\Gamma(\tau)\Gamma(\alpha)}$ |
| Moment Generating Function | Undefined |

### 0.1 Burr

Transformed Beta with $\tau=1$.

| | |
|---|---|
| Density function | $f(x)=\frac{\alpha\gamma\left(\frac{x}{\theta}\right)^\gamma}{x\left(1+\left(\frac{x}{\theta}\right)^\gamma\right)^{\alpha+1}}$ |
| Survival Function | $\frac{1}{\left(1+\left(\frac{x}{\theta}\right)^\gamma\right)^\alpha}$ |
| Mean | $\theta\frac{\Gamma\left(\alpha-\frac{1}{\gamma}\right)\Gamma\left(\frac{1}{\gamma}\right)}{\Gamma(\alpha)}$ |
| Raw Moments | $\mu_n'=\theta^n\frac{n\Gamma\left(\alpha-\frac{n}{\gamma}\right)\Gamma\left(\frac{n}{\gamma}\right)}{\Gamma(\alpha)}$ |
| Moment Generating Function | Undefined |

### Inverse Burr

Transformed Beta with $\alpha=1$.

| | |
|---|---|
| Density function | $f(x)=\frac{\tau\gamma\left(\frac{x}{\theta}\right)^{\gamma\tau}}{x\left(1+\left(\frac{x}{\theta}\right)^\gamma\right)^{\tau+1}}$ |
| Survival Function | $\frac{1}{\left(1+\left(\frac{x}{\theta}\right)^\gamma\right)^\alpha}$ |
| Mean | $\theta\frac{\Gamma\left(\tau+\frac{1}{\gamma}\right)\Gamma\left(1-\frac{1}{\gamma}\right)}{\Gamma(\tau)}$ |
| Raw Moments | $\mu_k'=\theta^k\frac{\Gamma\left(\tau+\frac{k}{\gamma}\right)\Gamma\left(1-\frac{k}{\gamma}\right)}{\Gamma(\tau)}$ |
| Moment Generating Function | Undefined |

### Generalised Pareto

Transformed Beta with $\gamma=1$.

| | |
|---|---|
| Density function | $f(x) = \left(\frac{\Gamma(\alpha+\tau)}{\Gamma(\alpha)\Gamma(\tau)}\right) \frac{\left(\frac{x}{\theta}\right)^\tau}{x\left(1+\left(\frac{x}{\theta}\right)\right)^{\alpha+\tau}}$ |
| Mean | $\theta \frac{\tau}{\alpha-1}$ |
| Raw Moments | $\mu'_k = \theta^k \frac{\Gamma(\tau+k)\Gamma(\alpha-k)}{\Gamma(\tau)\Gamma(\alpha)}$ |
| Moment Generating Function | Undefined |

## Pareto

Transformed Beta with $\tau = \gamma = 1$.

| | |
|---|---|
| Density function | $f(x) = \frac{\alpha}{\theta\left(1+\left(\frac{x}{\theta}\right)\right)^{\alpha+1}}$ |
| Survival Function | $\frac{1}{\left(1+\left(\frac{x}{\theta}\right)\right)^{\alpha}}$ |
| Mean | $\frac{\theta}{\alpha-1}$ (if $\alpha > 1$) |
| Variance | $\frac{\theta^2}{(\alpha-1)^2(\alpha-2)}$ (if $\alpha > 2$) |
| Raw Moments | $\mu'_k = \theta^k \frac{\Gamma(1+k)\Gamma(\alpha-k)}{\Gamma(\alpha)}$ |
| Moment Generating Function | Undefined |

## Inverse Pareto

Transformed Beta with $\alpha = \gamma = 1$.

| | |
|---|---|
| Density function | $f(x) = \frac{\tau\left(\frac{\theta}{x}\right)}{x\left(1+\left(\frac{\theta}{x}\right)\right)^{\tau+1}}$ |
| Survival Function | $1 - \frac{1}{\left(1+\left(\frac{\theta}{x}\right)\right)^{\tau}}$ |
| Mean | undefined |
| Moment Generating Function | Undefined |

## log-logistic

Transformed Beta with $\alpha = \tau = 1$.

| | |
|---|---|
| Density function | $f(x) = \frac{\gamma\left(\frac{x}{\theta}\right)^\gamma}{x\left(1+\left(\frac{x}{\theta}\right)^\gamma\right)^2}$ |
| Survival Function | $\frac{1}{\left(1+\left(\frac{x}{\theta}\right)^\gamma\right)}$ |
| Mean | $\theta\Gamma\left(1+\frac{1}{\gamma}\right)\Gamma\left(1-\frac{1}{\gamma}\right)$ |
| Raw Moments | $\mu'_k = \theta^k\Gamma\left(1+\frac{k}{\gamma}\right)\Gamma\left(1-\frac{k}{\gamma}\right)$ |
| Moment Generating Function | Undefined |

## Paralogistic

Transformed Beta with $\tau = 1$, $\alpha = \gamma$.

| | |
|---|---|
| Density function | $f(x) = \frac{\gamma\left(\frac{x}{\theta}\right)^\gamma}{x\left(1+\left(\frac{x}{\theta}\right)^\gamma\right)^{\gamma+1}}$ |
| Survival function | $S(x) = \frac{1}{\left(1+\left(\frac{x}{\theta}\right)^\gamma\right)^\gamma}$ |
| Mean | $\theta\frac{\Gamma\left(\gamma-\frac{1}{\gamma}\right)\Gamma\left(\frac{1}{\gamma}\right)}{\Gamma(\gamma)}$ |
| Raw Moments | $\mu'_k = \theta^k\frac{\Gamma\left(1+\frac{k}{\gamma}\right)\Gamma\left(\gamma-\frac{k}{\gamma}\right)}{\Gamma(\gamma)}$ |
| Variance | |
| Moment Generating Function | Undefined |

### Inverse Paralogistic

Transformed Beta with $\alpha = 1$, $\tau = \gamma$.

| | |
|---|---|
| Density function | $f(x) = \frac{\gamma\left(\frac{\theta}{x}\right)^{\gamma}}{x\left(1+\left(\frac{\theta}{x}\right)^{\gamma}\right)^{\gamma+1}}$ |
| Survival function | $S(x) = 1 - \frac{1}{\left(1+\left(\frac{\theta}{x}\right)^{\gamma}\right)^{\gamma}}$ |
| Mean | |
| Raw Moments | $\mu'_k = \theta^k \frac{\Gamma\left(\gamma+\frac{k}{\gamma}\right)\Gamma\left(1-\frac{k}{\gamma}\right)}{\Gamma(\gamma)}$ |
| Variance | |
| Excess loss | |
| Moment Generating Function | Undefined |

## Continuous Distributions: Transformed Gamma family

### Transformed Gamma

Limit of Transformed Beta as $\alpha \to \infty$ and $\theta \to \infty$ with $\alpha\theta^{\alpha} = \xi$.

| | |
|---|---|
| Density function | $f(x) = \frac{\tau\left(\frac{x}{\theta}\right)^{\tau\alpha} e^{-\left(\frac{x}{\theta}\right)^{\tau}}}{x\Gamma(\alpha)}$ |
| Mean | $\mu = \theta\frac{\tau\left(\alpha+\frac{1}{\tau}\right)}{\tau(\alpha)}$ |
| Raw moments | $\mu'_n = \theta^n \frac{\Gamma\left(\alpha+\frac{n}{\tau}\right)}{\Gamma(\alpha)}$ |

### Gamma

Transformed Gamma with $\tau = 1$

| | |
|---|---|
| Density function | $f(x) = \frac{\left(\frac{x}{\theta}\right)^{\alpha} e^{-\left(\frac{x}{\theta}\right)}}{x\Gamma(\alpha)}$ |
| Survival function (for $\alpha \in \mathbb{Z}^+$) | $S(x) = e^{-\frac{x}{\theta}}(1 + \cdots + \frac{\left(\frac{x}{\theta}\right)^{\alpha-1}}{(\alpha-1)!})$ |
| Mean | $\mu = \theta\alpha$ |
| Raw moments | $\mu'_n = \theta^n \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}$ |
| Variance | $\mu_n = \theta^n\alpha$ |
| Moment Generating Function | $M(t) = \frac{1}{(1-\theta t)^{\alpha}}$ |

### Weibull

Transformed Gamma with $\alpha = 1$

| | |
|---|---|
| Density function | $f(x) = \frac{\tau\left(\frac{x}{\theta}\right)^{\tau} e^{-\left(\frac{x}{\theta}\right)^{\tau}}}{x}$ |
| Survival function | $e^{-\left(\frac{x}{\theta}\right)^{\tau}}$ |
| Mean | $\mu = \theta\Gamma\left(1+\frac{1}{\tau}\right)$ |
| Raw moments | $\mu'_n = \theta^n\Gamma\left(1+\frac{n}{\tau}\right)$ |

### Exponential

Transformed Gamma with $\alpha = \tau = 1$

| | |
|---|---|
| Density function | $f(x) = \frac{e^{-\left(\frac{x}{\theta}\right)}}{\theta}$ |
| Survival function | $e^{-\frac{x}{\theta}}$ |
| Mean | $\mu = \theta$ |
| Raw moments | $\mu'_n = n!\theta^n$ |
| Variance | $\mu_n = \theta^n$ |
| Excess loss | $\theta e^{-\frac{x}{\theta}}$ |
| Moment Generating Function | $M(t) = \frac{1}{1-\theta t}$ |

### Inverse Transformed Gamma

Inverse of transformed gamma with $\theta = \frac{1}{\theta}$.

| | |
|---|---|
| Density function | $f(x) = \frac{\tau\left(\frac{\theta}{x}\right)^{\tau\alpha} e^{-\left(\frac{\theta}{x}\right)^{\tau}}}{x\Gamma(\alpha)}$ |
| Mean | $\mu = \theta\frac{\Gamma\left(\alpha-\frac{1}{\tau}\right)}{\Gamma(\alpha)}$ (if $\tau\alpha > 1$) |
| Raw moments | $\mu'_n = \theta^n\frac{\Gamma\left(\alpha-\frac{n}{\tau}\right)}{\Gamma(\alpha)}$ (if $\tau\alpha > n$) |

### Inverse Gamma

Inverse Transformed Gamma with $\tau = 1$. Inverse of gamma distribution with $\theta = \frac{1}{\theta}$.

| | |
|---|---|
| Density function | $f(x) = \frac{\left(\frac{\theta}{x}\right)^{\alpha} e^{-\left(\frac{\theta}{x}\right)}}{x\Gamma(\alpha)}$ |
| Survival function (for $\alpha \in \mathbb{Z}^+$) | $S(x) = 1 - e^{-\frac{\theta}{x}}(1 + \cdots + \frac{\left(\frac{\theta}{x}\right)^{\alpha-1}}{(\alpha-1)!})$ |
| Mean | $\mu = \frac{\theta}{\alpha-1}$ (if $\alpha > 1$) |
| Raw moments | $\mu'_n = \theta^n\frac{\Gamma(\alpha-n)}{\Gamma(\alpha)}$ (if $\alpha > n$) |
| Variance | $\mu_2 = \frac{\theta^2}{(\alpha-1)^2(\alpha-2)}$ |

### Inverse Weibull

Inverse Transformed Gamma with $\alpha = 1$. Inverse of Weibull distribution with $\theta = \frac{1}{\theta}$.

| | |
|---|---|
| Density function | $f(x) = \frac{\tau\left(\frac{\theta}{x}\right)^{\tau} e^{-\left(\frac{\theta}{x}\right)^{\tau}}}{x}$ |
| Survival function | $1 - e^{-\left(\frac{\theta}{x}\right)^{\tau}}$ |
| Mean | $\mu = \theta\Gamma\left(1 - \frac{1}{\tau}\right)$ (if $\tau > 1$) |
| Raw moments | $\mu'_n = \theta^n\Gamma\left(1 - \frac{n}{\tau}\right)$ (if $\tau > n$) |
| Moment Generating Function | Undefined |

### Inverse Exponential

Inverse Transformed Gamma with $\tau = \alpha = 1$, inverse of exponential with $\theta = \frac{1}{\theta}$.

| | |
|---|---|
| Density function | $f(x) = \frac{\theta e^{-\left(\frac{\theta}{x}\right)}}{x^2}$ |
| Survival function | $1 - e^{-\frac{\theta}{x}}$ |
| Mean | Undefined |

### Linear Exponential Family

| | |
|---|---|
| Density | $f_\theta(x) = \frac{p(x)e^{r(\theta)x}}{q(\theta)}$ |
| mean | $\mu(\theta) = \frac{q'(\theta)}{q(\theta)r'(\theta)}$ |
| Variance | $\mu_2(\theta) = \frac{\mu'(\theta)}{r'(\theta)}$ |

### Normal

| | |
|---|---|
| Density function | $f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ |
| Mean | $\mu = \mu$ |
| Variance | $\sigma^2$ |
| Moment Generating Function | $M(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$ |

### Uniform

| | |
|---|---|
| Density function | $f(x) = \frac{1}{b-a}$ (for $a < x < b$) |
| Survival funtion | $S(x) = \frac{b-x}{b-a}$ (for $a \leqslant x \leqslant b$) |
| Mean | $\mu = \frac{a+b}{2}$ |
| Variance | $\frac{(b-a)^2}{12}$ |
| Moment Generating Function | $M(t) = \frac{e^{bt} - e^{at}}{(b-a)t}$ |

## Discrete Distributions

### Binomial

| | |
|---|---|
| Probability | $p_k = \binom{n}{k} p^k (1-p)^{n-k}$ |
| mean | $\mu = np$ |
| raw moments | $\mathbb{E}(X \cdots (X+1-m)) = n \cdots (n+1-m)p^m$ |
| Variance | $\mu_2 = np(1-p)$ |
| p.g.f. | $P(z) = (1-p+pz)^n$ |
| $(a,b,0)$-class | $a = -\frac{p}{1-p}$, $b = \frac{(n-1)p}{1-p}$ |
| zero-truncated probability | $p_1^T = \frac{np(1-p)^{n-1}}{1-(1-p)^n}$ |

### Poisson

Limit of binomial as $n \to \infty$, $p \to 0$ with $np = \lambda$.

| | |
|---|---|
| Probability | $p_k = e^{-\lambda}\frac{\lambda^k}{k!}$ |
| mean | $\mu = \lambda$ |
| raw moments | $\mathbb{E}(X(X-1)\cdots(X+1-m)) = \lambda^m$ |
| Variance | $\mu_k = \lambda$ |
| p.g.f. | $P(z) = e^{\lambda(z-1)}$ |
| $(a,b,0)$-class | $a = 0$, $b = \lambda$ |
| zero-truncated probability | $p_1^T = \frac{\lambda e^{-\lambda}}{1-e^{-\lambda}}$ |

### Negative Binomial

- Gamma mixture of Poisson distributions where $\lambda$ follows a gamma distribution with $\theta = \beta$ and $\alpha = r$.
- Number of successes before $r$ failures if probability of success is $\frac{\beta}{1+\beta}$.
- Compound Poisson-Logarithmic distribution, where $\lambda = r\log\left(\frac{1}{1+\beta}\right)$ and $a = \frac{\beta}{1+\beta}$.

| | |
|---|---|
| Probability | $p_k = \binom{k+r-1}{k}\left(\frac{\beta}{1+\beta}\right)^k\left(\frac{1}{1+\beta}\right)^r$ |
| | $=\frac{r(r+1)\cdots(r+k-1)}{k!}\left(\frac{\beta}{1+\beta}\right)^k\left(\frac{1}{1+\beta}\right)^r$ |
| mean | $\mu = r\beta$ |
| Variance | $\mu_n = r\beta(1+\beta)\cdots(n-1+\beta)$ |
| p.g.f. | $P(z) = \left(\frac{1}{1+\beta-\beta z}\right)^r$ |
| $(a,b,0)$-class | $a = \frac{\beta}{1+\beta}$, $b = \frac{(r-1)\beta}{1+\beta}$ |
| zero-truncated probability | $p_1^T = \frac{r\beta}{(1+\beta)^{r+1}-(1+\beta)}$ |

### $(a,b,0)$ and $(a,b,1)$ Classes

$p_k = \left(a+\frac{b}{k}\right)p_{k-1}$ for $k > 1$ (and for $k > 0$ in the $(a,b,0)$ class).

| | |
|---|---|
| mean | $\mu = \frac{a+b}{1-a}$ |
| Variance | $\mu_2 = \frac{a+b}{(1-a)^2}$ |
| p.g.f. | $P(z) = \left(\frac{1-az}{1-a}\right)^{-\left(1+\frac{b}{a}\right)}$ |
| zero-truncated mean | $\mu = \frac{a+b}{(1-a)\left(1-(a+b)^{1+\frac{b}{a}}\right)}$ |
| zero-truncated probability | $p_1^T = \frac{a+b}{(1-a)^{-\left(1+\frac{b}{a}\right)}-1}$ |

### Logarithmic distribution

Negative binomial with $r = 0$. $(a,b,1)$-class with $a+b=0$, $a = \frac{\beta}{1+\beta}$.

| | |
|---|---|
| zero-truncated probability | $p_1^T = \frac{-a}{\log(1-a)} = \frac{\beta}{(1+\beta)\log(1+\beta)}$ |
| probability | $p_n = \frac{a^{n-1}}{n}p_1$ |
| mean | $\mu = \frac{-a}{(1-a)(\log(1-a))} = \frac{\beta}{\log(1+\beta)}$ |
| Variance | $\mu_2 = \frac{p_1-p_1^2}{(1-a)^2} = \frac{\beta(1+\beta)}{\log(1+\beta)} - \frac{\beta}{\log(1+\beta)^2}$ |
| p.g.f. | $P(z) = -\frac{\log(1-az)}{\log(1-a)}$ |
| $(a,b,1)$-class | $a = \frac{\beta}{1+\beta}$, $b = -\frac{\beta}{1+\beta}$ |

## Compound Distributions

### Moments:

Let the moments of the primary distribution be $\mu, \mu_2, \mu_3, \ldots$, and the moments of the secondary distribution by $\nu, \nu_2, \nu_3, \ldots$. The moments of the compound distribution are given by:

$\mu\nu$

$\mu\nu_2 + \mu_2\nu^2$

$\mu\nu_3 + \mu_2\nu\nu_2 + \mu_3\nu^3$.

### Recursive formula:

If the primary distribution is a member of the $(a,b,1)$-class, the probability mass function is defined as

$$f_S(k) = \frac{(p_1-(a+b)p_0)f_X(k) + \sum_{i=1}^{k}\left(a+\frac{bi}{k}\right)f_X(i)f_S(k-i)}{1-af_X(0)}$$

where:

- $f_X$ is the probability mass function of the secondary distribution
- $f_S$ is the probability mass function of the compound distribution
- $p_n$ is the probability that the primary distribution is $n$ (so $p_n = \left(a+\frac{b}{n}\right)p_{n-1}$)

## Non-parametric Estimators

### Greenwood's formula

$$\mathrm{Var}(S_n(y_j)) \approx S_n(y_j)^2\sum_{i=1}^{j}\frac{s_i}{r_i(r_i-s_i)}$$

where

- $y_i$ is the $i$th observed data point in increasing order.
- $s_i$ is the frequency of the observation $y_i$
- $r_i$ is the size of the risk set at observation $y_i$.

## Log-transformed Confidence intervals

$[S_n(x)^{\frac{1}{U}}, S_n(X)^U]$, where $U = e^{\Phi^{-1}\left(\frac{\alpha}{2}\right)\frac{\sigma}{S_n(x)\log(S_n(x))}}$.

- $\alpha$ is the confidence level (so for a 95% confidence interval, $\alpha = 0.05$).
- $\sigma$ is the standard deviation of the estimator $S_n(x)$.