ACSC/STAT 4703, Actuarial Models II
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Toby Kenney

Homework Sheet 1

Model Solutions

Basic Questions

1. *Loss amounts follow a gamma distribution with $\alpha = 3$ and $\theta = 8,000$. The distribution of the number of losses is given in the following table:*

Number of Losses	*Probability*
0	*0.04*
1	*0.43*
2	*0.34*
3	*0.19*

Assume all losses are independent and independent of the number of losses. The insurance company buys excess-of-loss reinsurance on the part of the loss above $100,000. Calculate the expected payment for this excess-of-loss reinsurance.

If there are n losses, the aggregate loss follows a gamma distribution with $\alpha = 3n$ and $\theta = 8000$. Conditional on this, the expected payment on the excess-of-loss reinsurance is given by

$$\int_{100000}^{\infty} x\frac{x^{\alpha-1}e^{-\frac{x}{\theta}}}{\theta^{\alpha}\Gamma(\alpha)}\,dx = \alpha\theta\int_{100000}^{\infty}\frac{x^{\alpha}e^{-\frac{x}{\theta}}}{\theta^{\alpha+1}\Gamma(\alpha+1)}\,dx$$

which is $\alpha\theta$ times the probability that a gamma distribution with parameters $\alpha+1$ and θ exceeds 100000. We can evaluate this directly with the `pgamma` function in `R`. Alternatively, we can compute it by integrating by parts, for example, for $\alpha = 3$:

$$\begin{aligned}
\int_{100000}^{\infty} x^3e^{-\frac{x}{\theta}}\,dx &= \left[-\theta x^3e^{-\frac{x}{\theta}}\right]_{100000}^{\infty} + 3\theta\int_{100000}^{\infty} x^2e^{-\frac{x}{\theta}}\,dx\\
&= \theta 100000^3e^{-\frac{100000}{\theta}} + 3\theta\int_{100000}^{\infty} x^2e^{-\frac{x}{\theta}}\,dx\\
&= 100000^3\theta e^{-\frac{100000}{\theta}} + 3\theta\left(100000^2\theta e^{-\frac{100000}{\theta}}\right) + 6\theta^2\int_{100000}^{\infty} xe^{-\frac{x}{\theta}}\,dx\\
&= e^{-\frac{100000}{\theta}}\left(100000^3\theta + 3\left(100000^2\theta^2\right) + 6\theta^2\left(100000\theta\right) + 6\theta^4\right)
\end{aligned}$$

Whichever method we use, we get the following table:

α	$\mathbb{E}((X-100000)_+)$
0	0
3	37.31
6	1659.23
9	14503.04

The expected payment on the excess-of-loss reinsurance is therefore

$$0.43 \times 37.31 + 0.34 \times 1659.23 + 0.19 \times 14503.04 = \$3,335.76$$

2. *An insurance company models loss frequency as negative binomial with $r = 2$, $\beta = 6$, and loss severity as pareto with $\alpha = 6$ and $\theta = \$30,000$. Calculate the expected aggregate payments if there is a policy limit of \$100,000 and a deductible of \$5,000 applied to each claim.*

   With the policy limit and deductible applied to each claim, the expected payment per loss is given by

$$\begin{aligned}\int_{5000}^{100000}\left(\frac{30000}{x+30000}\right)^6 dx &= \int_{35000}^{130000} 30000^6 u^{-6}\,du\\ &= 30000^6\left[-\frac{u^{-5}}{5}\right]_{35000}^{130000}\\ &= \frac{30000^6}{5}\left(\frac{1}{35000^5}-\frac{1}{130000^5}\right)\\ &= 2772.059\end{aligned}$$

   The expected number of losses is $2 \times 6 = 12$, so the expected aggregate loss is $12 \times 2772.059 = \$33,264.71$.

3. *Aggregate payments have a compund distribution. The frequency distribution is negative binomial with $r = 5$ and $\beta = 5$. The severity distribution is a Pareto distribution with $\alpha = 5$ and $\theta = 16000$. Use a Gamma approximation to aggregate payments to estimate the probability that aggregate payments are more than \$150,000.*

   The severity distribution has mean $\frac{16000}{5-1} = 4000$ and variance $\frac{16000^2}{(5-1)^2(5-2)} = \frac{16000000}{3}$. The negative binomial distribution has mean $5 \times 5 = 25$ and variance $5 \times 5 \times (5 + 1) = 150$. The mean and variance of the compound distribution are therefore $25 \times 4000 = 100000$ and

$$25 \times \frac{16000000}{3} + 150 \times 4000^2 = \frac{7600000000}{3}$$

   respectively. We find a gamma distribution with this mean and variance. Recall that the mean and variance of a gamma distribution are $\alpha\theta$ and

$\alpha\theta^2$ repectively. We therefore want to solve

$$\alpha\theta = 100000$$
$$\alpha\theta^2 = \frac{7600000000}{3}$$

The solution to this is $\theta = \frac{76000}{3}$ and $\alpha = \frac{300}{76}$. Now the probability that a gamma distribution with these parameters is more than 150,000 is 0.1522974.

4. *Claim frequency follows a negative binomial distribution with $r = 4$ and $\beta = 4.4$. Claim severity (in thousands) has the following distribution:*

Severity	*Probability*
1	*0.6*
2	*0.3*
3	*0.06*
4	*0.03*
5	*0.006*
6 or more	*0.003*

Use the recursive method to calculate the exact probability that aggregate claims are at least 6.

Aggregate claims are zero only if the number of claims is zero, which has probability $\left(\frac{1}{1+4.4}\right)^4 = 0.001176048$. The frequency distribution is from the $(a, b, 0)$-class with $a = \frac{4.4}{5.4} = \frac{22}{27}$ and $b = \frac{13.2}{5.4} = \frac{66}{27}$. The recurrence formula is

$$\begin{aligned} f_A(x) &= \frac{(p_1 - (a+b)p_0)f_S(x) + \sum_{m=1}^{x} \left(a + b\frac{m}{x}\right) f_S(m)f_A(x-m)}{1 - af_S(0)} \\ &= \sum_{m=1}^{x} \frac{22}{27}\left(1 + \frac{3m}{x}\right) f_S(m)f_A(x-m) \end{aligned}$$

Starting from $f_A(0) = 0.001176048$ we calculate the following values:

x	$f_A(x)$
0	0.001176048
1	0.002299827
2	0.003960814
3	0.005789318
4	0.007853989
5	0.010013262

The probability that the aggregate loss is at least 6 is therefore

$$1-0.001176048-0.002299827-0.003960814-0.005789318-0.007853989-0.010013262 = 0.9689067$$

R code:

```
fa<-rep(0.001176048,6)
fs<-c(.6,.3,.06,.03,.006)
fa[2]<-22/27*(1+3)*fa[1]*fs[1]
fa[3]<-22/27*sum((1+1.5*(1:2))*fa[2:1]*fs[1:2])
fa[4]<-22/27*sum((1+(1:3))*fa[3:1]*fs[1:3])
fa[5]<-22/27*sum((1+.75*(1:4))*fa[4:1]*fs[1:4])
fa[6]<-22/27*sum((1+.6*(1:5))*fa[5:1]*fs[1:5])
1-sum(fa)
```

5. *Use an arithmetic distribution ($h = 1$) to approximate a Pareto distribution with $\alpha = 6$ and $\theta = 30$.*

   (a) Using the method of rounding, calculate the mean of the the arithmetic approximation, conditional on lying in the interval 4.5 and 7.5. (That is, calculate $\mathbb{E}(X|4.5 < X < 5.5)$, where X follows the arithmetic distribution used to approximate.)

   Using the method of rounding, we have

$$p_5 = \left(\frac{30}{30+4.5}\right)^6 - \left(\frac{30}{30+5.5}\right)^6 = 0.06811295$$
$$p_6 = \left(\frac{30}{30+5.5}\right)^6 - \left(\frac{30}{30+6.5}\right)^6 = 0.05591691$$
$$p_7 = \left(\frac{30}{30+6.5}\right)^6 - \left(\frac{30}{30+7.5}\right)^6 = 0.04615374$$

   The conditional distribution is therefore

$$p_5 = \frac{0.06811295}{0.1701836} = 0.4002321$$
$$p_6 = \frac{0.05591691}{0.1701836} = 0.3285681$$
$$p_7 = \frac{0.04615374}{0.1701836} = 0.2711997$$

   The mean of this conditional distribution is therefore $0.4002321 \times 5 + 0.3285681 \times 6 + 0.2711997 \times 7 = 5.870968$.

(b) Using the method of local moment matching, matching 1 moment on each interval, estimate the probability that the value lies between 4.5 and 7.5.

Using the method of local moment matching, we select the probabilities p_{2n} and p_{2n+1} together to satisfy

$$\begin{aligned}
p_{2n} + p_{2n+1} &= P(2n - 0.5 < X \leqslant 2n + 1.5) \\
2np_{2n} + (2n+1)p_{2n+1} &= \mathbb{E}(X|2n - 0.5 < X \leqslant 2n + 1.5)P(2n - 0.5 < X \leqslant 2n + 1.5)
\end{aligned}$$

For the Pareto distribution we have

$$\begin{aligned}
P(2n - 0.5 < X \leqslant 2n + 1.5) &= \left(\frac{30}{30 + 2n - 0.5}\right)^6 - \left(\frac{30}{30 + 2n + 1.5}\right)^6 \\
\mathbb{E}(X|2n - 0.5 < X \leqslant 2n + 1.5)P(2n - 0.5 < X \leqslant 2n + 1.5) &= \int_{2n-0.5}^{2n+1.5} \frac{6 \times 30^6}{(30 + x)^7} x \, dx \\
&= \int_{2n+29.5}^{2n+31.5} 6 \times 30^6 u^{-7}(u - 30) \, dx \\
&= \int_{2n+29.5}^{2n+31.5} 6 \times 30^6 (u^{-6} - 30u^{-7}) \, dx \\
&= \left[30^7 u^{-6} - \frac{6}{5} \times 30^6 u^{-5}\right]_{2n+29.5}^{2n+31.5}
\end{aligned}$$

In particular for $n = 2$ and $n = 3$ we calculate

$$\begin{aligned}
P(3.5 < X \leqslant 5.5) &= 0.1515584 \\
\mathbb{E}(X|3.5 < X \leqslant 5.5)P(3.5 < X \leqslant 5.5) &= 0.671781 \\
P(5.5 < X \leqslant 7.5) &= 0.1020706
\end{aligned}$$

This gives us the equations

$$
\begin{aligned}
p_4 + p_5 &= 0.1515584 \\
4p_4 + 5p_5 &= 0.671781 \\
p_6 + p_7 &= 0.1020706 \\
p_5 &= 0.671781 - 4 \times 0.1515584 = 0.0655474 \\
p_5 + p_6 + p_7 &= 0.0655474 + 0.1020706 \\
&= 0.167618
\end{aligned}
$$

Standard Questions