ACSC/STAT 4703, Actuarial Models II
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Homework Sheet 4
Model Solutions

Basic Questions

1. *An insurance company models number of claims an individual makes in a year as following a negative binomial distribution with $\beta = 2.1$, and R an unknown parameter with prior distribution a gamma distribution with $\alpha = 4$ and $\theta = 0.01$.*

(a) What is the probability that a random individual makes exactly 4 claims?

The probability that an individual with parameter R makes exactly 4 claims is

$$\binom{R+3}{4}\left(\frac{1}{1+\beta}\right)^R\left(\frac{\beta}{1+\beta}\right)^4$$

The marginal probability of an individual making exactly 4 claims is found by integrating this over R:

$$\begin{aligned}
&\int_0^\infty \frac{10^8 r^3 e^{-100r}}{6}\binom{r+3}{4}\left(\frac{1}{3.1}\right)^r\left(\frac{2.1}{3.1}\right)^4\,dr\\
=&\frac{10^8}{6\times 24}\left(\frac{2.1}{3.1}\right)^4\int_0^\infty r^4(r+1)(r+2)(r+3)e^{-(100+\log(3.1))r}\,dr\\
=&\frac{10^8}{6\times 24}\left(\frac{2.1}{3.1}\right)^4\int_0^\infty (r^7+6r^6+11r^5+6r^4)e^{-(100+\log(3.1))r}\,dr\\
=&\frac{10^8}{6\times 24}\left(\frac{2.1}{3.1}\right)^4\left(\frac{7!}{(100+\log(3.1))^8}+\frac{6\times 6!}{(100+\log(3.1))^7}+\frac{11\times 5!}{(100+\log(3.1))^6}+\frac{6\times 4!}{(100+\log(3.1))^5}\right)\\
=&0.002177018
\end{aligned}$$

(b) The company now observes the following claim frequencies:

Number of claims	*Frequency*
0	*36*
1	*27*
2	*15*
3	*5*
4	*2*
5	*1*

What is the probability that $R > 0.4$? [You may use numerical integration to calculate this.]

The likelihood of the data given $R = r$ is

$$\frac{1}{3.1^{86r}}\left(\frac{2.1}{3.1}\right)^{27+15\times2+5\times3+2\times4+1\times5}\binom{r}{1}^{27}\binom{r+1}{2}^{15}\binom{r+2}{3}^{5}\binom{r+3}{4}^{2}\binom{r+4}{5}$$

This is proportional to

$$\frac{r^{50}(r+2)^{23}(r+3)^{8}(r+4)^{3}(r+5)^{1}}{3.1^{86r}}$$

The posterior distribution is therefore proportional to

$$e^{-100r}\frac{r^{53}(r+2)^{23}(r+3)^{8}(r+4)^{3}(r+5)}{3.1^{86r}}$$

The probability that $R > 0.4$ is therefore

$$\frac{\int_{0.6}^{\infty} e^{-(100+86\log(3.1))r}r^{53}(r+1)^{23}(r+2)^{8}(r+3)^{3}(r+4)}{\int_{0}^{\infty} e^{-(100+86\log(3.1))r}r^{53}(r+1)^{23}(r+2)^{8}(r+3)^{3}(r+4)} = 0.01901318$$

where the integrals were performed numerically.

(c) Calculate the predictive probability that this individual makes 5 claims next year. [You may use numerical integration to calculate this.]

The posterior density of R is proportional to

$$e^{-100r}\frac{r^{53}(r+1)^{23}(r+2)^{8}(r+3)^{3}(r+4)}{3.1^{86r}}$$

so the posterior probability that the individual makes 5 claims next year is

$$\frac{\left(\frac{2.1}{3.1}\right)^5\int_0^\infty 3.1^{-r}\binom{r+4}{5}e^{-100r}3.1^{-86r}r^{53}(r+1)^{23}(r+2)^{8}(r+3)^{3}(r+4)\,dr}{\int_0^\infty e^{-100r}3.1^{-86r}r^{53}(r+1)^{23}(r+2)^{8}(r+3)^{3}(r+4)\,dr}$$
$$=\left(\frac{2.1^5}{5!3.1^5}\right)\frac{\int_0^\infty e^{-100r}3.1^{-87r}r^{54}(r+1)^{24}(r+2)^{9}(r+3)^{4}(r+4)^{2}\,dr}{\int_0^\infty e^{-100r}3.1^{-86r}r^{53}(r+1)^{23}(r+2)^{8}(r+3)^{3}(r+4)\,dr}$$
$$=0.01115985$$

Where the integral was calculated numerically.

2. *An insurance company models loss sizes as following a Gamma distribution with $\alpha = 3$, and finds that the posterior distribution for Θ is an exponential distribution with $\theta = 1400$. Calculate the Bayes estimate for Θ based on a loss function:*

   (a) $l(\hat{\theta}, \theta) = (\hat{\theta} - \theta)^2$

   We are trying to choose $\hat{\theta}$ to minimise

$$\mathbb{E}\left((\hat{\theta}-\theta)^2\right) = \hat{\theta}^2 - 2\hat{\theta}\mathbb{E}(\theta) + \mathbb{E}(\theta^2)$$

For the exponential distribution, we have

$$\begin{aligned}
\mathbb{E}(\theta) &= 1400\\
\mathbb{E}(\theta^2) &= 2\times 1400^2\\
\mathbb{E}(\theta^3) &= 6\times 1400^3\\
\mathbb{E}(\theta^4) &= 24\times 1400^4
\end{aligned}$$

So we want to minimise $\hat{\theta}^2 - 2800\hat{\theta} + 3920000$. This is maximised by $\hat{\theta} = 1400$.

(b) $l(\hat{\theta},\theta) = (\hat{\theta}-\theta)^4$

We are trying to choose $\hat{\theta}$ to minimise

$$\begin{aligned}
\mathbb{E}\left((\hat{\theta}-\theta)^4\right) &= \hat{\theta}^4 - 4\hat{\theta}^3\mathbb{E}(\theta) + 6\hat{\theta}^2\mathbb{E}(\theta^2) - 4\hat{\theta}\mathbb{E}(\theta^3) + \mathbb{E}(\theta^4)\\
&= \hat{\theta}^4 - 4\times 1400\hat{\theta}^3 + 12\times 1400^2\hat{\theta}^2 - 24\times 1400^3\hat{\theta} + 24\times 1400^4\\
&= 1400^4\left(\left(\frac{\hat{\theta}}{1400}\right)^4 - 4\left(\frac{\hat{\theta}}{1400}\right)^3 + 12\left(\frac{\hat{\theta}}{1400}\right)^2 - 24\left(\frac{\hat{\theta}}{1400}\right) + 24\right)
\end{aligned}$$

This is minimised by $\frac{\hat{\theta}}{1400}$ is the solution to $4r^3 - 12r^2 + 24r - 24 = 0$, or equivalently $r^3 - 3r^2 + 6r - 6 = 0$. Numerically, we find the solution to this is 1.596071637983322, so the best estimate is $\hat{\theta} = 1400\times 1.596071637983322 = 2234.50$.

3. *An insurance company models annual claim frequencies as following a Poisson distribution with parameter Λ, where the prior distribution for Λ is a Gamma distribution with $\alpha = 3$ and $\theta = 0.06$. They observe a total of 5 claims in 15 years. Calculate a 95% credibility interval for Λ.*

   (a) Using an HPD interval.

   Since the Gamma distribution is a conjugate prior, the posterior distribution is a Gamma distribution with $\alpha = 3+5 = 8$ and $\theta = \frac{0.06}{1+0.06\times 15} = 0.031584210526$. The 95% credibility interval is the interval which under such a gamma distribution has probability 0.95, and such that the gamma density of the endpoints is equal. That is, if the integral is of the form $[l\theta, u\theta]$, then we have $l^{\alpha-1}e^{-l} = u^{\alpha-1}e^{-u}$ and $\int_l^u x^{\alpha-1}e^{-x}\,dx = 0.95\Gamma(\alpha)$.

   (b) With equal probability above and below the interval.

   The equal probability interval is between the 2.5th and 97.5th percentile of the gamma distribution, which is $[3.453832\times 0.031584210526, 14.42268\times 0.031584210526] = [0.1090684, 0.4554529]$.

4. *Calculate a conjugate prior distribution for the parameter α of a Pareto distribution.*

Suppose a Pareto distribution has fixed θ and α following a certain prior distibution. Suppose the observation is x. The likelihood of this observation with the given data is $\frac{\alpha}{x}\left(\frac{\theta}{\theta+x}\right)^{\alpha}$, which is proportional to $\alpha e^{-\log\left(1+\frac{x}{\theta}\right)\alpha}$, which is the density of a gamma distribution, so a gamma distribution is a conjugate prior.

Standard Questions

5. *An insurance company models number of claims made by an individual in a year as following a Poisson distribution where the parameter Λ follows a Gamma distribution with $\alpha = 3$ and $\theta = 0.06$. The company monitors the individual's claim history. If the individual's expected number of claims per year has decreased by 10% or more, the individual receives a discount on their premium. Suppose the individual's actual rate of claims is $\lambda = 0.25$, what is the probability that this individual ever receives a discount on their premium?*

If after N years, the number of claims is k, then the posterior distribution for λ is a Gamma distribution with $\alpha = 3 + k$ and $\theta = \frac{0.06}{1+0.06\times N}$. The expected number of claims under the prior distribution is $3 \times 0.06 = 0.18$. A decrease of 10% would reduce the expected number of claims per year to $0.18 \times 0.9 = 0.162$. The expected number of claims under the posterior distribution is $\frac{0.06(3+k)}{1+0.06N}$. The individual will therefore receive a discount if

$$\begin{aligned} \frac{0.06(3+k)}{1+0.06N} &\leqslant 0.162 \\ 0.18 + 0.06k &\leqslant 0.162 + 0.00972N \\ 0.00972N &\geqslant 0.018 + 0.06k \end{aligned}$$

We solve for the smallest value of N that entitles the policyholder to a discount for different values of k.

k	N
0	2
1	9
2	15
3	21
4	27
5	33
6	39
7	46
8	52
9	58
10	64

The probability that the individual makes no claims in the first two years is $e^{-0.5} = 0.6065307$. The probability that the individual makes at least one claim in the first two years and at most one claim within the first 9 years is $0.5e^{-0.5}e^{-1.75} = 0.05269961$ (They must make exactly one claim in the first

2 years and no claims in the following 7 years). The probability that the individual makes at least one claim within the first 2 years, at least 2 claims within the first 9 years, and at most 2 claims within the first 15 years is $0.5e^{-0.5}\times 1.75e^{-1.75}e^{-1.5}+e^{-0.5}\frac{0.5^2}{2}e^{-1.75}e^{-1.5}=0.02351775$. The probability that the individual makes at least 1 claim in the first 2 years, at least 2 claims in the first 9 years, at least 3 claims in the first 15 years, and at most 3 claims in the first 21 years is

$$e^{-5.25}\left(0.5\times 1.75\times 1.5+0.5\times\frac{1.75^2}{2}+\frac{0.5^2}{2}\times 1.75+\frac{0.5^2}{2}\times 1.5+\frac{0.5^3}{3}\right)=0.01314613$$

The probability that the individual makes at least 1 claim in the first 2 years, at least 2 claims in the first 9 years, at least 3 claims in the first 15 years, at least 4 claims in the first 21 years and at most 4 claims in the first 27 years is

$$\begin{aligned}e^{-6.75}&\left(0.5\times 1.75\times 1.5\times 1.5+0.5\times 1.75\times\frac{1.5^2}{2}+0.5\times\frac{1.75^2}{2}\times 1.5+0.5\times\frac{1.75^2}{2}\times 1.5+0.5\times\frac{1.75^3}{6}+\right.\\&\frac{0.5^2}{2}\times 1.5\times 1.5+\frac{0.5^2}{2}\times\frac{1.5^2}{2}+\frac{0.5^2}{2}\times 1.75\times 1.5+\frac{0.5^2}{2}\times 1.75\times 1.5+\frac{0.5^2}{2}\times\frac{1.75^2}{2}+\frac{0.5^3}{6}\times 1.5+\\&\left.\frac{0.5^3}{6}\times 1.5+\frac{0.5^3}{6}\times 1.75+\frac{0.5^4}{24}\right)=0.008275436\end{aligned}$$

The probability that the individual makes at least 1 claim in the first 2 years, at least 2 claims in the first 9 years, at least 3 claims in the first 15 years, at least 4 claims in the first 21 years, at least 5 claims in the first 27 years and at most 5 claims in the first 33 years is

$$\begin{aligned}e^{-8.25}&\left(0.5\times 1.75\times 1.5\times 1.5\times 1.5\left(1+\frac{3}{2}+\frac{1}{6}\right)+0.5\times\frac{1.75^2}{2}\times 1.5^2\left(3+\frac{3}{2}\right)+0.5\times\frac{1.75^3}{6}\times 1.5\times 3+\right.\\&0.5\times\frac{1.75^4}{24}+\frac{0.5^2}{2}\times 1.5^3\left(1+\frac{3}{2}+\frac{1}{6}\right)+\frac{0.5^2}{2}\times 1.75\times 1.5^2\left(3+\frac{3}{2}\right)+\frac{0.5^2}{2}\times\frac{1.75^2}{2}\times 1.5\times 3+\\&\frac{0.5^2}{2}\times\frac{1.75^3}{6}+\frac{0.5^3}{6}\times 1.5^2\left(3+\frac{3}{2}\right)+\frac{0.5^3}{6}\times 1.75\times 1.5\times 3+\frac{0.5^3}{6}\times\frac{1.75^2}{23}+\\&\left.\frac{0.5^4}{24}\times 1.5\times 3+\frac{0.5^4}{24}\times 1.75+\frac{0.5^5}{120}\right)=0.005896181\end{aligned}$$

The total probability of receiving the discount is therefore $0.6065307+0.05269961+0.02351775+0.01314613+0.008275436+0.005896181=0.71$.