ACSC/STAT 4703, Actuarial Models II
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Homework Sheet 1
Model Solutions

Basic Questions

1. *Aggregate payments have a compund distribution. The frequency distribution is negative binomial with $r = 2$ and $\beta = 2.5$. The severity distribution is an inverse gamma distribution with $\alpha = 2.5$ and $\theta = 15000$. Use a Pareto approximation to aggregate payments to estimate the probability that aggregate payments are more than \$150,000.*

   The mean and variance of the frequency, severity and aggregate loss distributions are given below:

	Mean	Variance
Frequency	$r\beta = 2 \times 2.5 = 5$	$r\beta(1+\beta) = 2 \times 2.5 \times 3.5 = 17.5$
Severity	$\frac{\theta}{\alpha-1} = \frac{15000}{1.5} = 10,000$	$\frac{\theta^2}{(\alpha-1)^2(\alpha-2)} = \frac{15000^2}{1.5^2 \times 0.5} = 200,000,000$
Aggregate Loss	$5 \times 10000 = 50,000$	$5 \times 200000000 + 17.5 * 10000^2 = 2,750,000,000$

   The mean of a Pareto distribution is $\frac{\theta}{\alpha-1}$ and the variance is $\frac{\theta^2\alpha}{(\alpha-1)^2(\alpha-2)}$.
   We therefore want to solve

$$\begin{aligned}
\frac{\theta}{\alpha-1} &= \quad 50,000 \\
\frac{\theta^2\alpha}{(\alpha-1)^2(\alpha-2)} &= 2,750,000,000 \\
\frac{\alpha}{\alpha-2} &= \frac{2,750,000,000}{50,000^2} = 1.1 \\
\alpha &= 1.1(\alpha-2) \\
0.1\alpha &= 2.2 \\
\alpha &= 22 \\
\theta &= 50000 \times 21 = 1,050,000
\end{aligned}$$

   The probability that the losses exceed \$150,000 is therefore

$$\frac{1}{\left(1+\frac{150000}{1050000}\right)^{22}} = 0.05298796$$

2. *Loss amounts follow a gamma distribution with $\alpha = 5$ and $\theta = 5,000$. The distribution of the number of losses is given in the following table:*

Number of Losses	*Probability*
0	*0.08*
1	*0.31*
2	*0.39*
3	*0.22*

Assume all losses are independent and independent of the number of losses. The insurance company buys excess-of-loss reinsurance on the part of the loss above \$100,000. Calculate the expected payment for this excess-of-loss reinsurance.

We can condition on the number of losses. If the number of losses is n, then the aggregate loss follows a gamma distribution with $\alpha = 5n$ and $\theta = 5,000$, so the expected payment on the excess of loss insurance is

$$\begin{aligned}\int_{100000}^{\infty}(x-100000)\frac{x^{5n-1}e^{-\frac{x}{5000}}}{5000^{5n}\Gamma(5n)}\,dx &= 5000\int_{20}^{\infty}\frac{(x-20)x^{5n-1}e^{-x}}{\Gamma(5n)}\,dx\\ &= 5000\left(\int_{20}^{\infty}\frac{x^{5n}e^{-x}}{\Gamma(5n)}\,dx - 20\int_{20}^{\infty}\frac{x^{5n-1}e^{-x}}{\Gamma(5n)}\,dx\right)\\ &= 5000\left(\left[-\frac{x^{5n}e^{-x}}{\Gamma(5n)}\right]_{20}^{\infty} + (5n-1-20)\int_{20}^{\infty}\frac{x^{5n-1}e^{-x}}{\Gamma(5n)}\,dx\right)\\ &= 5000\left(\frac{20^{5n}e^{-20}}{\Gamma(5n)} + (5n-1-20)\int_{20}^{\infty}\frac{x^{5n-1}e^{-x}}{\Gamma(5n)}\,dx\right)\end{aligned}$$

We evaluate this for different values of n

Number of Losses	Probability	Expected payment	Expected payment times probability
0	0.08	0	0
1	0.31	0.0185	0.0057
2	0.39	16.0677	6.2664
3	0.22	727.7356	160.1018

The total expected payment is therefore $0 + 0.0057 + 6.2664 + 160.1018 =$ \$166.37.

3. *An insurance company models loss frequency as binomial with $n = 84$, $p = 0.14$, and loss severity as inverse exponential with $\theta = 20,000$. Calculate the expected aggregate payments if there is a policy limit of \$50,000 and a deductible of \$10,000 applied to each claim.*

With the deductible and policy limit, the expected claim amount **per loss** is

$$\begin{aligned}
\int_{10000}^{50000} (x-10000)\frac{20000e^{-\frac{20000}{x}}}{x^2}\,dx &= 20000\int_{10000}^{50000} \frac{e^{-\frac{20000}{x}}}{x}\,dx - 2\times 10^8\int_{10000}^{50000} \frac{e^{-\frac{20000}{x}}}{x^2}\,dx\\
&= 20000\int_{0.4}^{2} \frac{e^{-u}}{u}\,du - 10000\int_{0.4}^{2} e^{-u}\,du\\
&= 10000\int_{0.16}^{4} e^{-\sqrt{v}}\,dv - 10000\int_{0.4}^{2} e^{-u}\,du\\
&= 20000\left[-\left(1+\sqrt{v}\right)e^{-\sqrt{v}}\right]_{0.16}^{4} + 10000\left[e^{-u}\right]_{0.4}^{2}\\
&= 20000\left(1.4e^{-0.4}-3e^{-2}\right) + 10000\left(e^{-2}-e^{-0.4}\right)\\
&= 10000\left(1.8e^{-0.4}-5e^{-2}\right)\\
&= 5298.997
\end{aligned}$$

The expected aggregate payments are given by expected loss frequency times expected claim per loss, which is $84\times 0.14\times 5298.997 = \$62,316.20$.

4. *Claim frequency follows a negative binomial distribution with $r = 2$ and $\beta = 8.5$. Claim severity (in thousands) has the following distribution:*

Severity	*Probability*
1	*0.2*
2	*0.5*
3	*0.18*
4	*0.07*
5 or more	*0.05*

Use the recursive method to calculate the exact probability that aggregate claims are at least 5.

The severity distribution is zero truncated, so the aggregate loss is zero only if there are no losses, which has probability $\frac{1}{9.5^2} = 0.01108033$. The primary distribution is a negative binomial with $r = 2$, $\beta = 8.5$. This is from the $(a, b, 0)$ class with $a = \frac{\beta}{1+\beta} = \frac{17}{19}$ and $b = \frac{(r-1)\beta}{1+\beta} = \frac{17}{19}$. The recurrence is then

$$f_S(x) = \sum_{y=1}^{x} \frac{17}{19}\left(1+\frac{y}{x}\right)\frac{1}{9.5^2}\binom{y+1}{y}\left(\frac{8.5}{9.5}\right)^{y} f_S(x-y)$$

We use this recurrence to find $f_S(1)$, $f_S(2)$, $f_S(3)$ and $f_S(4)$:

$$f_S(1) = \frac{17}{19} \times 2 \times \frac{4}{19^2} \times 2 \times \frac{17}{19} \times 0.01108033 = 0.000393148$$

$$f_S(2) = \frac{17}{19} \times \frac{4}{19^2}\left(1.5 \times 2 \times \frac{17}{19} \times 0.000393148 + 2 \times 3 \times \left(\frac{17}{19}\right)^2 \times 0.01108033\right) = 0.0005381082$$

$$f_s(3) = \frac{17}{19} \times \frac{4}{19^2}\left(\frac{4}{3} \times 2 \times \frac{17}{19} \times 0.0005381082 + \frac{5}{3} \times 3 \times \left(\frac{17}{19}\right)^2 \times 0.000393148 \right.$$
$$\left. +2 \times 4 \times \left(\frac{17}{19}\right)^3 \times 0.01108033\right) = 0.0006578026$$

$$f_S(4) = 0.0007573641$$

The probability that the aggregate loss is at least 5 is therefore $1 - 0.01108033 - 0.000393148 - 0.0005381082 - 0.0007573641 = 0.987231$.

5. *Use an arithmetic distribution ($h = 1$) to approximate a Pareto distribution with $\alpha = 3$ and $\theta = 40$.*

(a) Using the method of rounding, calculate the mean of the arithmetic approximation.

Under the method of rounding, $p_0 = 1 - \left(\frac{40}{40.5}\right)^3$, and for $n \geqslant 1$, we have $p_n = \left(\frac{40}{39.5+n}\right)^3 - \left(\frac{40}{40.5+n}\right)^3$. Now the mean of the arithmetic distribution is

$$\begin{aligned}\sum_{n=1}^{\infty} np_n &= \sum_{n=1}^{\infty} n\left(\frac{40}{39.5+n}\right)^3 - \left(\frac{40}{40.5+n}\right)^3 \\ &= \sum_{n=1}^{\infty}\left(\frac{40}{39.5+n}\right)^3 \\ &= 19.99688\end{aligned}$$

where the sum was computed by truncating the sum at $n = 1000000$. By comparing with the integral, we have $\sum_{n=1000001}^{\infty}\left(\frac{40}{39.5+n}\right)^3 \approx \int_{1000000.5}^{\infty}\left(\frac{40}{39.5+x}\right)^3 dx = \int_{1000040}^{\infty} \frac{40^3}{u^3}\, du = 40^3\left[-\frac{u^{-2}}{3}\right]_{1000040}^{\infty} = \frac{40^3}{3\times 1000040^2} =$.

(b) Using the method of local moment matching, matching 1 moment on each interval, estimate the probability that the value is larger than 20.5.

To match one moment on each interval, we need the interval to contain two points, so the intervals are $[0, 1.5)$ and $[2n - 0.5, 2n + 1.5)$. Since the probabilities are matched for each interval, the probability that the value is larger than 19.5 is the same as the probability that the pareto distribution is larger than 19.5, which is $\left(\frac{40}{59.5}\right)^3 =$. We just need to calculate p_{20}. We have the equations for p_{20} and p_{21}:

$$\begin{aligned}
p_{20} + p_{21} &= \left(\frac{40}{59.5}\right)^3 - \left(\frac{40}{61.5}\right)^3 = 0.02868833 \\
20p_{20} + 21p_{21} &= \int_{19.5}^{21.5} x\frac{3\times 40^3}{(40+x)^4}\,dx \\
&= 3\times 40^3 \int_{59.5}^{61.5} \frac{u-40}{u^4}\,dx \\
&= 3\times 40^3 \int_{59.5}^{61.5} u^{-3} - 40u^{-4}\,dx \\
&= 3\times 40^3 \left[\frac{40u^{-3}}{3} - \frac{u^{-2}}{2}\right]_{59.5}^{61.5} \\
&= \frac{40^4}{61.5^3} - \frac{40^4}{59.5^3} + \frac{3\times 40^3}{2\times 59.5^2} - \frac{3\times 40^3}{2\times 61.5^2} \\
&= 0.5874786 \\
p_{20} &= 21\left(\left(\frac{40}{59.5}\right)^3 - \left(\frac{40}{61.5}\right)^3\right) - \left(\frac{40^4}{61.5^3} - \frac{40^4}{59.5^3} + \frac{3\times 40^3}{2\times 59.5^2} - \frac{3\times 40^3}{2\times 61.5^2}\right) \\
&= 61\left(\left(\frac{40}{59.5}\right)^3 - \left(\frac{40}{61.5}\right)^3\right) - \frac{3\times 40^3}{2\times 59.5^2} + \frac{3\times 40^3}{2\times 61.5^2} \\
&= 0.01497636
\end{aligned}$$

so $P(X > 20.5) = \left(\frac{40}{59.5}\right)^3 - 0.01497636 = 0.2888525$.

Standard Questions

6. *The number of claims an insurance company receives follows a negative binomial distribution with $r = 160$ and $\beta = 14$. Claim severity follows a negative binomial distribution with $r = 5$ and $\beta = 1.2$. Calculate the probability that aggregate losses exceed \$17,000.*

   (a) Starting the recurrence 6 standard deviations below the mean. [You need to calculate the recurrence up to $f_s(20,000)$.]

   For the frequency distribution, we have $a = \frac{\beta}{1+\beta} = \frac{14}{15}$ and $b = \frac{(r-1)\beta}{1+\beta} = \frac{2226}{15}$. The recurrence is therefore

$$f_s(x) = \frac{\sum_{y=1}^{x} \frac{14}{15}\left(1 + 159\frac{y}{x}\right)\frac{1}{2.2^5}\binom{y+4}{y}\left(\frac{1.2}{2.2}\right)^y f_s(x-y)}{1 - \frac{14}{15\times 2.2^5}}$$

The mean of the aggregate loss distribution is $160\times14\times5\times1.2 = 13440$ and the variance is $160\times14\times5\times1.2\times2.2+160\times14\times15\times(5\times1.2)^2 = 1239168$. Six standard deviations below the mean is therefore $13440-6\sqrt{1239168} = 6760.925$. We will therefore start the recurrence at $x = 6761$. We set $f_s(6760) = 0$ and $f_s(6761) = 1$.

```
f<-rep(0,200000)
f[6761]<-1
```

Now we apply the recurrence

```
for(x in 6762:200000){
    temp<-0
    for(y in 1:(x-6761)){
        temp<-temp+14/15*(1+159*y/x)*1/2.2^5*choose(y+4,4)*(1.2/2.2)^y*f[x-y]
    }
    f[x]<-temp/(1-14/15/2.2^5)
}
```

Finally, we rescale and calculate the probability

```
f<-f/sum(f)
probability=sum(f[17001:20000])
```

This gives the answer 0.001402609

(b) Using a suitable convolution.

We note that the primary distribution is the sum of 8 negative binomial distributions with $r = 20$ and $\beta = 44$. For these primary distributions, the recurrence is

$$f_s(x) = \frac{\sum_{y=1}^{x} \frac{44}{45}\left(1+19\frac{y}{x}\right)\frac{1}{2.8^{21}}\binom{y+20}{y}\left(\frac{1.8}{2.8}\right)^y f_s(x-y)}{1-\frac{44}{45\times2.8^{21}}}$$

and we have $f_s(0) = (1+44(1-\frac{1}{2.8^{21}}))^{-20} =$.

We therefore apply the recurrence:

```
g<-rep(0,10000)
g[1]=(1+14*(1-2.2^(-5)))^(-20)

for(x in 2:10000){
    temp<-0
    for(y in 1:(x-1)){
        temp<-temp+14/15*(1+19*y/x)*1/2.2^5*choose(y+4,4)*(1.2/2.2)^y*f[x-y]
```

```
    }
    g[x]<-temp/(1-14/15/2.2^5)
}
```

We then define the convolution function:

```
ConvolveSelf<-function(n){
  convolution<-vector("numeric",2*length(n))
  for(i in 1:(length(n))){
    convolution[i]<-sum(n[1:i]*n[i:1])
  }
  for(i in 1:(length(n))){
    convolution[2*length(n)+1-i]<-sum(n[length(n)+1-(1:i)]*n[length(n)+1-(i:1)])
  }
  return(convolution)
}
```

Finally we convolve the function with itself three times:

```
g2<-convolveself(g)
g4<-convolveself(g2)
g8<-convolveself(g4)

probabilityconvolve<-sum(g8[17002:30000])
```

Note that the sum is taken from 17002, because g[1] corresponds to $f_S(0)$.
This gives the answer 0.001402713.