ACSC/STAT 4703, Actuarial Models II
Fall 2018
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Homework Sheet 1
Model Solutions

Basic Questions

1. *Aggregate payments have a compund distribution. The frequency distribution is negative binomial with $r = 5$ and $\beta = 1.3$. The severity distribution is a Weibull distribution with $\tau = 2.3$ and $\theta = 15000$. Use a Pareto approximation to aggregate payments to estimate the probability that aggregate payments are more than $150,000.*

   The negative binomial distribution has mean $5 \times 1.3 = 6.5$ and variance $5 \times 1.3 \times 2.3 = 14.95$. The Weibull distribution has mean $15000 \times \Gamma\left(1 + \frac{1}{2.3}\right) = 13288.72$ and variance $15000^2 \left(\Gamma\left(1 + \frac{2}{2.3}\right) - \Gamma\left(1 + \frac{1}{2.3}\right)^2\right) = 37542739$. The mean aggregate payment is therefore $6.5 \times 13288.72 = \$86,376.68$ and the variance is $6.5 \times 37542739 + 14.95 \times 13288.72^2 = 2,884,049,488$.

   To approximate the distribution by a Pareto distribution, we get the parameters by solving:

$$\begin{aligned}
\frac{\theta}{\alpha - 1} &= 86376.68 \\
\frac{\alpha\theta^2}{(\alpha-1)^2(\alpha-2)} &= 2884049488 \\
\frac{\alpha - 2}{\alpha} &= \frac{86376.68^2}{2884049488} = 2.58696 \\
\frac{2}{\alpha} &= -1.58696 \\
\alpha &= -1.26027121036 \\
\theta &= 86376.68 \times -2.26027121036 = -195234.72305
\end{aligned}$$

   [Due to an error in the quesiton, this is not actually an acceptable Pareto distribution. However, we will proceed assuming that it can just be used in this way.]

   The probability that the aggregate payments exceed $150,000 is therefore $\left(\frac{-195234.72305}{150000-195234.72305}\right)^{-1.26027121036} = 0.158350887147$.

2. *Loss amounts follow a gamma distribution with $\alpha = 2$ and $\theta = 12,000$. The distribution of the number of losses is given in the following table:*

Number of Losses	*Probability*
0	*0.17*
1	*0.21*
2	*0.37*
3	*0.25*

Assume all losses are independent and independent of the number of losses. The insurance company buys excess-of-loss reinsurance on the part of the loss above \$100,000. Calculate the expected payment for this excess-of-loss reinsurance.

If the number of losses is k, then the aggregate loss distribution is a gamma distribution with $\alpha = 2k$ and $\theta = 12000$. The expected payment on this excess-of-loss insurance is therefore

$$\int_{100000}^{\infty}(x-100000)\frac{x^{2k-1}e^{-\frac{x}{12000}}}{12000^{2k}\Gamma(2k)}\,dx$$
$$=\int_{100000}^{\infty}\frac{x^{2k}e^{-\frac{x}{12000}}}{12000^{2k}\Gamma(2k)}\,dx-100000\int_{100000}^{\infty}\frac{x^{2k-1}e^{-\frac{x}{12000}}}{12000^{2k}\Gamma(2k)}\,dx$$
$$=24000k\int_{100000}^{\infty}\frac{x^{2k}e^{-\frac{x}{12000}}}{12000^{2k+1}\Gamma(2k+1)}\,dx-100000\int_{100000}^{\infty}\frac{x^{2k-1}e^{-\frac{x}{12000}}}{12000^{2k}\Gamma(2k)}\,dx$$

The integrals in the final expression are the probability that a gamma distribution with $\theta = 12000$ and shape parameters $\alpha = 2k+1$ and $\alpha = 2k$ respectively exceed 100000. These can easily be computed in `R` using the `pgamma` function. For example, when $k = 1$ we get

```
> pgamma(100000/12000,shape=2+1,lower.tail=FALSE)
[1] 0.01058961
> pgamma(100000/12000,shape=2,lower.tail=FALSE)
[1] 0.002243448
```

We get the following expected payments on the excess-of-loss reinsurance.

No. of claims	Expected Aggregate loss payment
0	0
1	$24000 \times 0.01058961 - 100000 \times 0.002243448 = 29.80582$
2	$48000 \times 0.082072946 - 100000 \times 0.033773395 = 562.1619$
3	$72000 \times 0.274376713 - 100000 \times 0.162572197 = 3497.904$

The total expected aggregate loss payment is therefore $0.21 \times 29.80582 + 0.37 \times 562.1619 + 0.25 \times 3497.904 = \$1,088.74$

3. *An insurance company models loss frequency as binomial with $n = 95$, $p = 0.12$, and loss severity as Pareto with $\theta = 20,000$ and $\alpha = 1.5$. Calculate the expected aggregate payments if there is a policy limit of \$50,000 and a deductible of \$10,000 applied to each claim.*

With a policy limit of \$50,000 and a deductible of \$10,000, the expected payment for each loss is

$$\begin{aligned}\int_{10000}^{50000} \left(\frac{20000}{20000+x}\right)^{1.5} dx &= 20000 \int_{1.5}^{3.5} u^{-1.5}\, du \\ &= 20000 \left[-2u^{-0.5}\right]_{1.5}^{3.5} \\ &= 40000 \left(\frac{1}{\sqrt{1.5}} - \frac{1}{\sqrt{3.5}}\right) \\ &= 11278.9638842\end{aligned}$$

The expected number of losses is $95 \times 0.12 = 11.4$, so the expected aggregate payment is $11.4 \times 11278.9638842 = \$128,580.19$.

4. *Claim frequency follows a negative binomial distribution with $r = 3$ and $\beta = 5.9$. Claim severity (in thousands) has the following distribution:*

Severity	*Probability*
1	*0.3*
2	*0.45*
3	*0.14*
4	*0.08*
5 or more	*0.03*

Use the recursive method to calculate the exact probability that aggregate claims are at least 5.

The recursive method uses the formula

$$f_S(x) = \frac{(p_1 - (a+b)p_0)f_X(x) + \sum_{i=1}^{x} \left(a + \frac{bi}{x}\right) f_X(i) f_S(x-i)}{1 - af_X(0)}$$

Since claim frequency follows a negative binomial distribution, we have $a = \frac{\beta}{1+\beta} = \frac{5.9}{6.9}$ and $b = (r-1)\frac{\beta}{1+\beta} = \frac{11.8}{6.9}$. We also have $p_1 - ap_0 = 0$, and the severity distribution is zero truncated, so the denominator is 1. This allows us to simplify the recurrence to

$$f_S(x) = \frac{5.9}{6.9} \sum_{i=1}^{x} \left(1 + \frac{2i}{x}\right) f_X(i) f_S(x-i)$$

Since the severity distribution is zero-truncated, the aggregate loss is zero only if the number of claims is zero, which has probability $\frac{1}{6.9^3} = 0.003044056632$. The recurrence therefore gives

$$
\begin{aligned}
f_S(1) &= \frac{5.9}{6.9} \times 3 \times 0.3 \times 0.003044 = 0.00234260010376 \\
f_S(2) &= \frac{5.9}{6.9} (2 \times 0.3 \times 0.002343 + 3 \times 0.45 \times 0.003044) = 0.00471575586104 \\
f_S(3) &= \frac{5.9}{6.9} \left(\frac{5}{3} \times 0.3 \times 0.004716 + \frac{7}{3} \times 0.45 \times 0.002343 + 3 \times 0.14 \times 0.003044\right) = 0.00521261735752 \\
f_S(4) &= \frac{5.9}{6.9} (1.5 \times 0.3 \times 0.005213 + 2 \times 0.45 \times 0.004716 \\
&\qquad +2.5 \times 0.14 \times 0.002343 + 3 \times 0.08 \times 0.003044) = 0.00696102521739
\end{aligned}
$$

The probability that the aggregate loss is at least 5 is therefore

$$1 - 0.003044 - 0.002343 - 0.004716 - 0.005213 - 0.006961 = 0.977724$$

5. *Use an arithmetic distribution ($h = 1$) to approximate a Weibull distribution with $\tau = 3$ and $\theta = 20$.*

   (a) Using the method of rounding, calculate the mean of the arithmetic approximation.

   For $n \geqslant 1$, the probability p_n for the arithmetic approximation is

$$p_n = P(n - 0.5 < X < n + 0.5) = e^{-\left(\frac{n-0.5}{20}\right)^3} - e^{-\left(\frac{n+0.5}{20}\right)^3}$$

   This gives us

$$S(n) = e^{-\left(\frac{n-0.5}{20}\right)^3}$$

   . Numerically evaluating this sum, the mean of this distribution is therefore

$$\sum_{n=1}^{\infty} e^{-\left(\frac{n-0.5}{20}\right)^3} = 17.85959$$

   (b) Using the method of local moment matching, matching 1 moment on each interval, estimate the probability that the value is larger than 14.5.

   Under the method of local moment matching, we choose p_{2n} and p_{2n+1} so that

$$
\begin{aligned}
P(2n - 0.5 < X < 2n + 1.5) &= p_{2n} + p_{2n+1} \\
\mathbb{E}(X | 2n - 0.5 < X < 2n + 1.5) &= \frac{2np_{2n} + (2n+1)p_{2n+1}}{p_{2n} + p_{2n+1}}
\end{aligned}
$$

   From the first equation, we have $p_0 + \ldots + p_{13} = P(X < 13.5) = 1 - e^{-\left(\frac{13.5}{20}\right)^3} = 0.264751598029$, so we just need to find p_{14}.

We have

$$p_{14}+p_{15} = P(13.5 < X < 15.5) = e^{-\left(\frac{13.5}{20}\right)^3} - e^{-\left(\frac{15.5}{20}\right)^3} = 0.107417476049$$

and

$$\begin{aligned}14p_{14}+15p_{15} &= P(13.5 < X < 15.5)\mathbb{E}(X|13.5 < X < 15.5)\\ &= 13.5e^{-\left(\frac{15.5}{20}\right)^3} + \int_{13.5}^{15.5} e^{-\left(\frac{x}{20}\right)^3}\,dx - 15.5e^{-\left(\frac{15.5}{20}\right)^3}\\ &= 1.559665\end{aligned}$$

where the integral was calculated numerically.

This allows us to solve

$$p_{14} = 15\times 0.107417476049 - 1.559665 = 0.05159714074$$

Therefore

$$P(X > 14.5) = 1 - 0.264751598029 - 0.05159714074 = 0.683651261231$$

Standard Questions

6. *The number of claims an insurance company receives follows a negative binomial distribution with $r = 110$ and $\beta = 17$. Claim severity follows a negative binomial distribution with $r = 9$ and $\beta = 0.8$. Calculate the probability that aggregate losses exceed \$15,000.*

   (a) Starting the recurrence 6 standard deviations below the mean [You need to calculate the recurrence up to $f_s(20,000)$.]

   The mean aggregate loss is $110\times 17\times 9\times 0.8 = 13464$ and the variance of the aggregate loss is $110\times 17\times 18\times (9\times 0.8)^2 + 110\times 17\times 9\times 0.8\times 1.8 = 1769169.6$. Six standard deviations below the mean is therefore $13464 - 6\sqrt{1769169.6} = 5483.39$, so we will start our recurrence at $x = 5483$. The recurrence is

$$f_S(x) = \frac{\sum_{i=1}^{x}\frac{17}{18}\left(1+\frac{109i}{x}\right)\binom{i+8}{i}\frac{1}{1.8^9}\left(\frac{0.8}{1.8}\right)^i f_s(x-i)}{1-\frac{17}{18\times 1.8^9}}$$

   We start with $f_S(5482) = 0$ and $f_S(5483) = 1$ and use this recurrence to obtain all values of f_S up to $f_S(40000)$.

```
f<-rep(0,40000)
f[5483]<-1

for(i in seq_len(40000-5483)){
fX<-choose((1:i)+8,8)*(4/9)^(1:i)*(5/9)^9
   fS[5483+i]=17/18/(1-17/18/1.8^9)*sum((1+109*(1:i)/i)*fS[5483+i-(1:i)]*fX)
}
```

We then standardise f_S by dividing by its sum and evaluate the probability:

```
fS<-fS/sum(fS)
sum(fS[15001:40000])
```

This gives the probability 0.1257334.

(b) Using a suitable convolution.

We have that a negative binomial distribution with parameters $r = 110$, $\beta = 17$ is a sum of 8 i.i.d. negative binomial distributions with parameters $r = 13.75$ and $\beta = 17$. We can therefore use the recurrence to get the distribution where the loss frequency distribution is negative binomial with $r = 13.75$ and $\beta = 17$. The P.G.F. of the negative binomial is $P(z) = (1 + \beta - \beta z)^{-r}$ The probability that this distribution is zero is

$$P(f_x(0)) = \left(1 + 17 - 17\left(\frac{1}{1.8^9}\right)\right)^{-13.75} = 5.86817672559 \times 10^{-18}$$

From this, we compute the recurrence

$$f_S(x) = \frac{\sum_{i=1}^{x} \frac{17}{18}\left(1 + \frac{12.75i}{x}\right)\binom{i+8}{i}\frac{1}{1.8^9}\left(\frac{0.8}{1.8}\right)^i f_s(x-i)}{1 - \frac{17}{18\times 1.8^9}}$$

```
g<-rep(0,15001)
g[1]=(1+17-17/1.8^9)^(-13.75)

for(x in 2:15001){
    y<-seq_len(x-1)
    temp<-sum((1+12.75*y/(x-1))*choose(y+8,8)*(4/9)^y*g[x-y])
    g[x]<-temp*14/15/(1-17/18/1.8^9)/1.8^9
}
```

Having computed $f_S(x)$ for $x = 0, \ldots, 15000$, we convolve this with itself 3 times to get the aggregate loss distribution

```
ConvolveSelf<-function(n){
  l<-length(n)
  convolution<-vector("numeric",2*l)
  for(i in seq_len(l)){
    convolution[i]<-sum(n[1:i]*n[i:1])
  }
  for(i in 1:(length(n))){
    convolution[2*l+1-i]<-sum(n[l+1-(1:i)]*n[l+1-(i:1)])
  }
  return(convolution)
}
```

```
g2<-ConvolveSelf(g)
g4<-ConvolveSelf(g2)
g8<-ConvolveSelf(g4)

1-sum(g8[1:15001])
```

This gives the probability value 0.1257334.