ACSC/STAT 4703, Actuarial Models II
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Homework Sheet 4
Model Solutions

Basic Questions

1. *An insurance company sells marine insurance. It estimates that the standard deviation of the aggregate annual claim is \$7,603 and the mean is \$1,324.*

   (a) How many years history are needed for an individual or group to be assigned full credibility? (Use $r = 0.02$, $p = 0.95$.)

   For an individual with n year's of credibility, this individuals average aggregate claims are approximately normally distributed with mean 1324 and standard deviation $\frac{7,603}{\sqrt{7,603}}$. The probability that the relative error in this value is more than 0.02 is therefore the probability that a normal distribution with mean 1324 and standard deviation $\frac{7603}{\sqrt{n}}$ is outside the interval $1324 \pm 0.02 \times 1324 = [1297.52, 1350.48]$ The probability of this is $2\Phi\left(\frac{26.48\sqrt{n}}{7603}\right) - 1$. The standard for full credibility is n such that

$$\begin{aligned}
2\Phi\left(\frac{26.48\sqrt{n}}{7603}\right) - 1 &= 0.95 \\
\Phi\left(\frac{26.48\sqrt{n}}{7603}\right) &= 0.975 \\
\frac{26.48\sqrt{n}}{7603} &= 1.959964 \\
\sqrt{n} &= 562.749482326 \\
n &= 316687
\end{aligned}$$

   The standard premium for this policy is \$1,324. A company has claimed a total of \$48,300 in the last 8 years.

   (b) What is the Credibility premium for this company, using limited fluctuation credibility?

   The credibility of the company's experience is $\sqrt{\frac{8}{316687}} = 0.00502608553829$. The company's average annual claims are $\frac{48300}{8} = \$6,037.50$, so the credibility premium is $0.00502608553829 \times 6037.50 + 0.994973914462 \times 1324 = \$1347.69$.

2. *A workers' compensation insurance company classifies employees as high, medium or low risk. Annual claims from high risk employees follow a*

Pareto distribution with $\alpha = 4$ and $\theta = 9000$. Annual claims from medium risk employees follow a Gamma distribution with $\alpha = 4$ and $\theta = 500$. Annual claims from low risk employees follow a normal distribution with mean \$1,300 and standard deviation \$900. 25% of employees are high risk, 60% are medium risk and 15% are low risk.

(a) Calculate the expectation and variance of the aggregate annual claims from a randomly chosen employee.

The expected aggregate annual claims are $0.25 \times \frac{9000}{4-1} + 0.6 \times 4 \times 500 + 0.15 \times 1300 = 2145$. The variance is

$$0.25 \times \frac{9000}{(4-1)^2(4-2)} + 0.6 \times 4 \times 500^2 + 0.15 \times 900^2$$
$$+ 0.25 \times \left(\frac{9000}{4-1} - 2145\right) + 0.6 \times (4 \times 500 - 2145) + 0.15 \times (1300 - 2145)$$

=721625

(b) Given that an employee's annual claims over the past 4 years are \$6,000, \$50 and \$1,400, what are the expectation and variance of the employee's claims next year?

For a high-risk employee, the likelihood of these claims is

$$4\frac{9000^4}{(9000+6000)^5} \times 4\frac{9000^4}{(9000+50)^5} \times 4\frac{9000^4}{(9000+1400)^5} = 3.22273205974 \times 10^{-12}$$

For a medium risk employee, the likelihood is

$$\frac{6000^3 e^{-\frac{6000}{500}}}{500^4 \times 6} \times \frac{50^3 e^{-\frac{50}{500}}}{500^4 \times 6} \times \frac{1400^3 e^{-\frac{1400}{500}}}{500^4 \times 6} = 4.74970118141 \times 10^{-16}$$

For a low risk employee, the likelihood is

$$\frac{e^{-\frac{(6000-1300)^2}{2\times 900^2}}}{\sqrt{2\pi} \times 900} \times \frac{e^{-\frac{(50-1300)^2}{2\times 900^2}}}{\sqrt{2\pi} \times 900} \times \frac{e^{-\frac{(1400-1300)^2}{2\times 900^2}}}{\sqrt{2\pi} \times 900} = 3.94901116609 \times 10^{-17}$$

The posterior probability of each type of employee is therefore

High risk:

$$\frac{0.25 \times 3.222732 \times 10^{-12}}{0.25 \times 3.222732 \times 10^{-12} + 0.6 \times 4.749701 \times 10^{-16} + 0.15 \times 3.949011 \times 10^{-17}} = 0.999639063274$$

Medium risk:

$$\frac{0.6 \times 4.749701 \times 10^{-16}}{0.25 \times 3.222732 \times 10^{-12} + 0.6 \times 4.749701 \times 10^{-16} + 0.15 \times 3.949011 \times 10^{-17}} = 0.000353587211234$$

Low risk:

$$\frac{0.15 \times 3.949011 \times 10^{-17}}{0.25 \times 3.222732 \times 10^{-12} + 0.6 \times 4.749701 \times 10^{-16} + 0.15 \times 3.949011 \times 10^{-17}} = 0.00000734951416951$$

The expected claim in the following year is therefore

$$0.999639063274 \times 3000 + 0.000353587211234 \times 2000 + 0.00000734951416951 \times 1300 = 2999.63391861$$

The variance is

$$\begin{aligned}
&0.999639 \times 4500000 + 0.000353587 \times 1000000 + 0.00000734951 \times 810000 \\
&\quad + 0.999639(3000 - 2999.63)^2 + 0.000353587(2000 - 2999.63)^2 + 0.0000073495(1300 - 2999.63)^2 \\
=&4499110.01835
\end{aligned}$$

Standard Questions

3. *For a certain insurance policy, the book premium is based on average claim frequency of 2.6 claims per year, and average claim severity of \$7,200. The standard for full credibility is 40 policy years for claim frequency and 110 claims for severity. The insurance company wants to change the standard for full credibility to 45 policy years for aggregate claims. A particular group has 100 claims in 32 policy years of history. For what values of total claims would the change in standards result in an increased premium for this group?*

   If the group has 100 claims, then the credibility of the group's average severity is $Z = \sqrt{\frac{100}{110}} = 0.953462589246$. The credibility of the groups claim frequency is $\sqrt{\frac{32}{40}} = 0.894427191$. The expected number of claims for the next year is therefore $0.894427191 \times \frac{100}{32} + 0.105572809 \times 2.6 = 3.06957427528$. If the average claim amount is x, then the expected claim amount next year is $\sqrt{\frac{100}{110}}x + 7200\left(1 - \sqrt{\frac{100}{110}}\right) = 0.953462589246x + 335.069357429$. Therefore the expected aggregate claims are

$$3.06957427528(0.953462589246x + 335.069357429) = 2.92672423639x + 1028.52028$$

   Under the new policy, the average aggregate claims is $\frac{100x}{32}$ and the credibility is $Z = \sqrt{\frac{32}{45}} = 0.843274042712$. Therefore the credibility premium is

$$2.63523138348x + 0.156725957288 \times 2.6 \times 7200 = 2.63523138348x + 2933.90992043$$

   Therefore the premium will increase if

$$\begin{aligned}
2.63523138348x + 2933.90992043 &> 2.92672423639x + 1028.52028 \\
0.29149285291x &< 1905.38964043 \\
x &< 6536.65989203
\end{aligned}$$

4. *Aggregate claims for an individual are believed to follow a gamma distribution with $\alpha = 4$ and θ varying between individuals. For a randomly chosen individual, theta follows an inverse gamma distribution with $\alpha = 3$ and $\theta = 500$. The insurance company uses limited fluctuation credibility with $r = 0.05$ and $p = 0.95$ to determine an individual's premium. If an individual has 8 years of past history, for what value of total claims during these 8 years would the limited fluctuation credibility premium equal the fair premium (using the Bayesian method)?*

The average aggregate claim is 2000, and the variance is $4\mathbb{E}(\Theta^2)+16\operatorname{Var}(\Theta) = 4\times\frac{1500^2}{3\times 2}+16\times\frac{1500^2}{3^2\times 2} = 3500000$. For the average claim over n policy-years past history, the mean is \$2,000 and the variance is $\frac{3500000}{n}$. The standard for full credibility is given by

$$\begin{aligned}
2\Phi\left(\frac{2000\times 0.05\sqrt{n}}{\sqrt{3500000}}\right)-1 &= 0.95\\
\frac{2000\times 0.05\sqrt{n}}{\sqrt{3500000}} &= 1.959964\\
\sqrt{n} &= 1.959964\sqrt{350}\\
n &= 1344.51060846
\end{aligned}$$

The credibility is therefore $Z = \sqrt{\frac{8}{1344.51060846}} = 0.0771370235389$. meaning that the premium is $0.0771370235389\frac{A}{8} + 0.922862976461\times 2000 = 0.00964212794236A + 1845.72595292$.

Using the Bayesian approach, we have that the aggregate claims over 8 years follow a gamma distribution with $\alpha = 32$. If the aggregate claims are A, then the likelihood for a given value of θ is therefore $\frac{A^{31}e^{-\frac{A}{\Theta}}}{\Theta^{32}\Gamma(32)}$. The predictive distribution therefore has density proportional to

$$\begin{aligned}
\int_0^\infty \frac{A^{31}e^{-\frac{A}{\theta}}}{\theta^{32}}\theta^{-4}e^{-\frac{1500}{\theta}}\frac{x^3e^{-\frac{x}{\theta}}}{\theta^4}\,d\theta &= \int_0^\infty \frac{x^3e^{-\frac{A+x+1500}{\theta}}}{\theta^{40}}\,d\theta\\
&\propto \frac{x^3}{(A+1500+x)^{39}}\\
&= \frac{(B+x)^3-3B(B+x)^2+3B^2(B+x)-B^3}{(B+x)^{39}}\\
&= (B+x)^{-36}-3B(B+x)^{-37}+3B^2(B+x)^{-38}-B^3(B+x)^{-39}
\end{aligned}$$

where $B = A+1500$. This is a linear combination of Pareto densities:

$$\frac{1}{35}\times 35B^{35}(B+x)^{-36}-\frac{3}{36}\times 36B^{36}(B+x)^{-37}+\frac{3}{37}\times 37B^{37}(B+x)^{-38}-\frac{1}{38}\times 38B^{38}(B+x)^{-39}$$

The expected payment is therefore

$$\frac{\frac{1}{35}\frac{B}{34}-\frac{3}{36}\frac{B}{35}+\frac{3}{37}\frac{B}{36}-\frac{1}{38}\frac{B}{37}}{\frac{1}{35}-\frac{3}{36}+\frac{3}{37}-\frac{1}{38}}=\frac{0.000000398452410836B}{0.00000338684549211}=0.117647058823B$$

The difference between expected payments is therefore

$$0.117647058823(A+1500)-(0.00964212794236A+1845.72595292)=0.108004930881A-1669.25536469$$

. The two methods therefore agree when the aggregate losses are

$$\frac{1669.25536469}{0.108004930881}=\$15455.36$$

5. *An insurance company has 4 years of past history on a Tennants insurance policy, denoted X_1, X_2, X_3, X_4. It uses a formula $\hat{X_5}=\alpha_0+\alpha_1X_1+\alpha_2X_2+\alpha_3X_3+\alpha_4X_4$ to calculate the credibility premium in the fifth year. It has the following information on the policy:*

   - *In year 1, the expected aggregate claim was \$300.*
   - *Expected aggregate claims increase by 3% per year.*
   - *The coefficient of variation of the aggregate claims is 0.7 in every year.*
   - *The correlation (recall $Corr(X,Y)=\frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}}$) between aggregate claims in years i and j is $\frac{5-|i-j|}{25}$ for all $i\neq j$.*

   Find a set of equations which can determine the values of α_0, α_1, α_2, α_3 and α_4. [You do not need to solve these equations.]

   We have $\mathbb{E}(X_i)=300(1.03)^{i-1}$ and $\mathrm{Var}(X_i)=0.7^2\times 300^2(1.03)^{2(i-1)}$. Finally, for $i\neq j$ we get

$$\begin{aligned}\mathrm{Cov}(X_i,X_j)&=\frac{5-|i-j|}{25}\sqrt{0.7^2\times 300^2(1.03)^{2(i-1)}0.7^2\times 300^2(1.03)^{2(j-1)}}\\&=\frac{5-|i-j|}{25}0.7^2\times 300^2(1.03)^{i+j-2)}\end{aligned}$$

Recall that the greatest accuracy credibility equations give us

$$\begin{aligned}\mathbb{E}(X_5)&=\alpha_0+\alpha_1\mathbb{E}(X_1)+\alpha_2\mathbb{E}(X_2)+\alpha_3\mathbb{E}(X_3)+\alpha_4\mathbb{E}(X_4)\\\mathrm{Cov}(X_5,X_i)&=\alpha_1\,\mathrm{Cov}(X_1,X_i)+\alpha_2\,\mathrm{Cov}(X_2,X_i)+\alpha_3\,\mathrm{Cov}(X_3,X_i)+\alpha_4\,\mathrm{Cov}(X_4,X_i)\end{aligned}$$

Substituting the values we have gives

$$\begin{aligned}
300(1.03)^4 &= \alpha_0 + 300\alpha_1 + 300(1.03)\alpha_2 + 300(1.03)^2\alpha_3 + 300(1.03)^3\alpha_4 \\
\frac{1}{25}(1.03)^4 &= \alpha_1 + \frac{4}{25}(1.03)^1\alpha_2 + \frac{3}{25}(1.03)^2\alpha_3 + \frac{2}{25}(1.03)^3\alpha_4 \\
\frac{2}{25}(1.03)^5 &= \frac{4}{25}(1.03)^1\alpha_1 + (1.03)^2\alpha_2 + \frac{4}{25}(1.03)^3\alpha_3 + \frac{3}{25}(1.03)^3\alpha_4 \\
\frac{3}{25}(1.03)^5 &= \frac{3}{25}(1.03)^2\alpha_1 + \frac{4}{25}(1.03)^3\alpha_2 + (1.03)^4\alpha_3 + \frac{4}{25}(1.03)^4\alpha_4 \\
\frac{4}{25}(1.03)^6 &= \frac{2}{25}(1.03)^3\alpha_1 + \frac{3}{25}(1.03)^4\alpha_2 + \frac{4}{25}(1.03)^5\alpha_3 + (1.03)^6\alpha_4
\end{aligned}$$

[We can simplify and solve these equations reasonably straightforwardly:

$$\begin{aligned}
300 &= \alpha_0(1.03)^{-4} + 300\alpha_1(1.03)^{-4} + 300(1.03)^{-3}\alpha_2 + 300(1.03)^{-2}\alpha_3 + 300(1.03)^{-1}\alpha_4 \\
1 &= 25(1.03)^{-4}\alpha_1 + 4(1.03)^{-3}\alpha_2 + 3(1.03)^{-2}\alpha_3 + 2(1.03)^{-1}\alpha_4 \\
2 &= 4(1.03)^{-4}\alpha_1 + 25(1.03)^{-3}\alpha_2 + 4(1.03)^{-2}\alpha_3 + 3(1.03)^{-1}\alpha_4 \\
3 &= 3(1.03)^{-4}\alpha_1 + 4(1.03)^{-3}\alpha_2 + 25(1.03)^{-2}\alpha_3 + 4(1.03)^{-1}\alpha_4 \\
4 &= 2(1.03)^{-4}\alpha_1 + 3(1.03)^{-3}\alpha_2 + 4(1.03)^{-2}\alpha_3 + 25(1.03)^{-1}\alpha_4
\end{aligned}$$

Letting $\beta_i = \alpha_i(1.03)^{i-5}$ these become

$$\begin{aligned}
1 &= \frac{\alpha_0(1.03)^{-4}}{300} + \beta_1 + \beta_2 + \beta_3 + \beta_4 \\
1 &= 25\beta_1 + 4\beta_2 + 3\beta_3 + 2\beta_4 \\
2 &= 4\beta_1 + 25\beta_2 + 4\beta_3 + 3\beta_4 \\
3 &= 3\beta_1 + 4\beta_2 + 25\beta_3 + 4\beta_4 \\
4 &= 2\beta_1 + 3\beta_2 + 4\beta_3 + 25\beta_4
\end{aligned}$$

$$\beta_1 = 0.01061653$$
$$\beta_2 = 0.04737300$$
$$\beta_3 = 0.08886678$$
$$\beta_4 = 0.13924723$$
$$\alpha_0 = 241.049$$
$$\alpha_1 = 0.01194900$$
$$\alpha_2 = 0.05176576$$
$$\alpha_3 = 0.09427877$$
$$\alpha_4 = 0.14342465$$

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