ACSC/STAT 4703, Actuarial Models II
WINTER 2020
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Midterm Examination
Model Solutions

Each part question (a, b, c, etc.) is worth 1 mark. You should have been provided with a formula sheet. **No other notes are permitted**. Scientific calculators are permitted, but not graphical calculators.

Here are some values of the Gamma distribution function with $\theta = 1$ that may be needed for this examination:

x	α	$F(x)$	x	α	$F(x)$	x	α	$F(x)$
245	255	0.2697208	2.5	4	0.2424239	4.375	4	0.6361773
$\left(\frac{7.5}{12}\right)^3$	$\frac{4}{3}$	0.1117140	3.875	3	0.7430029	4.875	4	0.7169870
$\left(\frac{9.5}{12}\right)^3$	$\frac{4}{3}$	0.2507382	4.375	3	0.8118663	5.375	4	0.7837292
2.5	1	0.917915	4.875	3	0.8644174	2.156	5	0.06782354
2.5	2	0.7127025	5.375	3	0.9035828	3.203	5	0.219922
2.5	3	0.4561869	3.875	4	0.5417358	8.542	5	0.9274742

1. *Using an arithmetic distribution ($h = 1$) to approximate a Weibull distribution with $\tau = 2$ and $\theta = 7$, calculate the probability that the value is more than 6.5, for the approximation using the method of local moment matching, matching 1 moment on each interval.*

   We have

$$\int_a^b \frac{2}{49}x^2 e^{-\left(\frac{x}{7}\right)^2}\,dx = \left[-xe^{-\left(\frac{x}{7}\right)^2}\right]_a^b + \int_a^b e^{-\left(\frac{x}{7}\right)^2}\,dx$$

$$= ae^{-\left(\frac{a}{7}\right)^2} - be^{-\left(\frac{b}{7}\right)^2} + 7\sqrt{\pi}\int_a^b \frac{e^{-\frac{x^2}{2\left(\frac{7}{\sqrt{2}}\right)^2}}}{\sqrt{2\pi}\frac{7}{\sqrt{2}}}\,dx$$

$$= ae^{-\left(\frac{a}{7}\right)^2} - be^{-\left(\frac{b}{7}\right)^2} + 7\sqrt{\pi}\left(\Phi\left(\frac{\sqrt{2}b}{7}\right) - \Phi\left(\frac{\sqrt{2}a}{7}\right)\right)$$

   In particular,

$$\int_{5.5}^{7.5} \frac{2}{49}x^2 e^{-\left(\frac{x}{7}\right)^2}\,dx = 1.435464$$

$$\int_{6}^{7} \frac{2}{49}x^2 e^{-\left(\frac{x}{7}\right)^2}\,dx = 0.7254893$$

The intervals are $[0,1.5]$, $[1.5,3.5]$, $[3.5,5.5]$, $[5.5,7.5]$, etc. This means that $P(A > 5.5) = P(X > 5.5) = e^{-\left(\frac{5.5}{7}\right)^2} = 0.5393735$. It just remains to calculate p_6. We have

$$\begin{aligned}
p_6 + p_7 &= P(5.5 < X < 7.5) = e^{-\left(\frac{5.5}{7}\right)^2} - e^{-\left(\frac{7.5}{7}\right)^2} = 0.2220899 \\
6p_6 + 7p_7 &= \int_{5.5}^{7.5} x \left(\frac{2x}{7^2}\right) e^{-\left(\frac{x}{7}\right)^2}\,dx \\
&= \frac{2}{49}\int_{5.5}^{7.5} x^2 e^{-\left(\frac{x}{7}\right)^2}\,dx \\
&= 1.435464
\end{aligned}$$

This gives $p_6 = 7 \times 0.2220899 - 1.435464 = 0.1191653$ so $P(A > 6.5) = 0.5393735 - 0.1191653 = 0.4202082$.

Using overlapping intervals, we have $P(X > 6) = e^{-\left(\frac{6}{7}\right)^2} = 0.4796523$, while

$$\begin{aligned}
p_6 + p_7 &= e^{-\left(\frac{6}{7}\right)^2} - e^{-\left(\frac{7}{7}\right)^2} = 0.1117728 \\
6p_6 + 7p_7 &= \int_{6}^{7} x \left(\frac{2x}{7^2}\right) e^{-\left(\frac{x}{7}\right)^2}\,dx \\
&= \frac{2}{49}\int_{6}^{7} x^2 e^{-\left(\frac{x}{7}\right)^2}\,dx \\
&= 0.7254893
\end{aligned}$$

Thus $p_6 = 7 \times 0.1117728 - 0.7254893 = 0.0569203$. This gives $P(A > 6.5) = 0.4796523 - 0.0569203 = 0.422732$.

2. *Claim frequency follows a Poisson distribution with $\lambda = 2.4$. Claim severity (in thousands) has the following distribution:*

Severity	*Probability*
0	*0.21*
1	*0.44*
2	*0.32*
3	*0.03*

The company buys excess-of loss reinsurance for aggregate losses exceeding 2.

(a) Use the recursive method to calculate the probability that the reininsurance makes a payment.

Firstly we have $f_S(0) = P_S(0) = P_F(P_X(0)) = P_F(f_X(0)) = e^{2.4(0.21-1)} = 0.1501681$.

The recurrence is

$$f_S(n) = \sum_{m=1}^{n} 2.4\frac{m}{n} f_X(m) f_S(n-m)$$

This gives

$$f_S(1) = 2.4 \times 0.44 \times 0.1501681 = 0.1585775136$$
$$f_S(2) = 2.4\left(\frac{1}{2} \times 0.44 \times 0.1585775136 + 0.32 \times 0.1501681\right) = 0.199058027981$$

Thus the probability that the aggregate loss is more than 2 is $1-0.1501681-0.1585775136 - 0.199058027981 = 0.492196358419$.

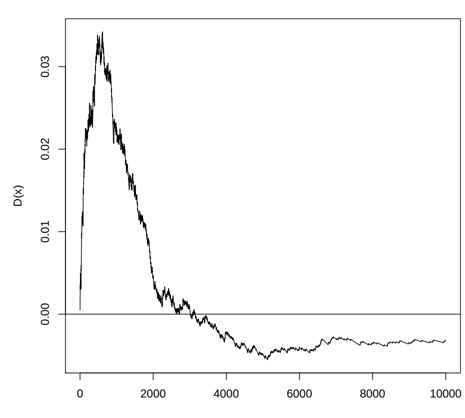
(b) What is the expected payment on the reinsurance? [Hint: Consider the difference between the expected aggregate losses payments and the expected payments made with reinsurance.]

The expected loss amount is $0\times 0.21+1\times 0.44+2\times 0.32+3\times 0.03 = 1.17$. The expected aggregate loss without reinsurance is therefore $2.4 \times 1.17 = 2.808$.

With reinsurance, the expected aggregate loss paid by the insurance is $0.1585775136\times 1+0.199058027981\times 2+0.492196358419\times 3 = 2.03328264482$.

The expected reinsurance payment is therefore $2.808 - 2.03328264482 = 0.77471735518$.

3. *An insurance company collects a sample of 6000 claims. Based on previous experience, it believes these claims might follow a Pareto distribution with $\alpha = 3$ and $\theta = 2000$. To test this, it computes the following plot of $D(x) = F_n(x) - F^*(x)$.*



(a) How many of the claims in their sample were more than 3,000?

From the graph we see that $D(3000) \approx 0$. For the Pareto distribution, we have $F^*(3000) = 1 - \left(\frac{2000}{2000+3000}\right)^3 = 0.936$. This gives $F_n(3000) \approx 0.936$, so there are approximately $6000 \times 0.936 = 5616$ samples below 3,000.

(b) Which of the following statements best describes the fit of the Pareto distribution to the data:

(i) The Pareto distribution assigns too much probability to high values and too little probability to low values.

(ii) The Pareto distribution assigns too much probability to low values and too little probability to high values.

(iii) The Pareto distribution assigns too much probability to tail values and too little probability to central values.

(iv) The Pareto distribution assigns too much probability to central values and too little probability to tail values.

Justify your answer.

We see that $D(x) > 0$ for $x < 3000$ and $D(x) < 0$ for $x > 3000$. This means that $F^*(x) < F_n(x)$ for small x, and $F^*(x) > F_n(x)$, so $S^*(x) < S_n(x)$ for large x. Thus the model assigns too little probability to tail values, and too much to central values, so (iv) best describes the fit.

4. *An insurance company collects the following sample:*

   21.23 23.88 83.10 86.25 226.15 381.31 458.78 606.75 1201.73 1857.35

   They model this as following a distribution with the following distribution function:

x	$F(x)$	$\log(F(x_i)) - \log(F(x_{i+1}))$	$\log(1-F(x_{i+1})) - \log(1-F(x_i))$
21.23	*0.07957669*	*0.20005185*	*0.01933289*
23.88	*0.09720023*	*1.35724389*	*0.37202795*
83.10	*0.37766854*	*0.02503241*	*0.01550247*
86.25	*0.38724181*	*0.46555216*	*0.46950458*
226.15	*0.61683496*	*0.14798137*	*0.29673049*
381.31	*0.71521477*	*0.04070322*	*0.11018515*
458.78	*0.74492690*	*0.05233498*	*0.17068346*
606.75	*0.78495083*	*0.09206273*	*0.43385253*
1201.73	*0.86064646*	*0.03942506*	*0.28548762*
1857.35	*0.89525524*	*0.11064642*	*NA*

   Calculate the Kolmogorov-Smirnov statistic for this model and this data.

   The largest difference happens when $x = 222.15^-$, when $F_n(x) = 0.4$ and $F^*(x) = 0.61683496$. The Kolmogorov-Smirnov statistic is therefore $0.61683496 - 0.4 = 0.21683496$.

   We can compare the possible values in a table:

x	$F(x)$	$F_n(x^-)$	$F_n(X^+)$	$D(x)$
21.23	0.07957669	**0.0**	0.1	0.07957669
23.88	0.09720023	0.1	**0.2**	0.10279977
83.10	0.37766854	**0.2**	0.3	0.17766854
86.25	0.38724181	**0.3**	0.4	0.08724181
226.15	0.61683496	**0.4**	0.5	**0.21683496**
381.31	0.71521477	**0.5**	0.6	0.21521477
458.78	0.74492690	**0.6**	0.7	0.14492690
606.75	0.78495083	**0.7**	0.8	0.08495083
1201.73	0.86064646	**0.8**	0.9	0.06064646
1857.35	0.89525524	0.9	**1.0**	0.10474476

5. *An insurance company collects a sample of 700 claims. They want to decide whether this data is better modeled as following an inverse exponential, or a transformed beta distribution. After calculating MLE estimates for the parameters (1 parameter for the inverse exponential and 4 for the transformed beta), log-likelihoods for the two distributions are:*

Distribution	*log-likelihood*
Inverse Exponential	*-4341.82*
Transformed Beta	*-4334.55*

Use the Bayes Information Criterion (BIC) to decide which distribution is a better fit for the data.

The BIC for the inverse exponential is given by $-4341.82 - 1 \times \frac{\log(700)}{2} = -4345.09554017$. The BIC for the transformed beta is given by $-4334.55 - 4 \times \frac{\log(700)}{2} = -4347.65216067$.

The inverse exponential has higher BIC, so it is a better fit for the data.

6. *A homeowner's house is valued at \$340,000. However, the home is insured only to a value of \$190,000. The insurer requires 70% coverage for full insurance. The home sustains \$8,000 of damage from a break-in. The deductible is \$4,000, decreasing linearly to zero for losses of \$10,000. How much does the insurer reimburse?*

The value required for full insurance is $3400000 \times 0.7 = 2380000$. The proportion of insurance paid is therefore $\frac{190000}{238000} = 0.798319327731$. For a loss of \$8,000, the deductable on full coverage is $4000 \times \frac{10000-8000}{10000-4000} = 1333.33333333$. Therefore the amount paid is $(8000 - 1333.33333333) \times 0.798319327731 = \$5,322.13$.

7. *The following table shows the cumulative losses (in thousands) on claims from one line of business of an insurance company over the past 4 years.*

Accident year	*Development year*			
	0	*1*	*2*	*3*
2016	*645*	*1021*	*1098*	*1307*
2017	*729*	*1100*	*1123*	
2018	*804*	*1210*		
2019	*751*			

Using the mean for calculating loss development factors, esimate the total reserve needed for payments to be made in 2020 using the Bornhuetter-Fergusson method. The expected loss ratio is 0.72 and the earned premiums in each year are given in the following table:

Year	*Earned Premiums (000's)*
2016	*1857*
2017	*1944*
2018	*2143*
2019	*2095*

[Assume no more payments are made after development year 3.]

The mean loss development factors are

Year	loss development factors
0/1	$\frac{3331}{2178} = 1.52938475666$
1/2	$\frac{2221}{2121} = 1.0471475719$
2/3	$\frac{1307}{1098} = 1.19034608379$

This means that we can calculate the expected proportion of total losses paid in the first n years:

Year	Cumulative Proportion of total losses paid	Proportion of total losses paid
0	$\frac{1}{1.19034608379 \times 1.0471475719 \times 1.52938475666} = 0.524568375947$	0.524568375947
1	$\frac{1}{1.19034608379 \times 1.0471475719} = 0.802266877999$	0.277698502052
2	$\frac{1}{1.19034608379} = 0.840091813312$	0.037824935313
3	1	0.159908186688

The expected total losses for each accident year are

Year	Earned Premiums (000's)	Expected losses
2017	1944	$1944 * 0.72 = 1399.68$
2018	2143	$2143 * 0.72 = 1542.96$
2019	2095	$2095 * 0.72 = 1508.40$

This gives us the following expected total losses

Accident Year	Expected Losses	Reserves Year 1	Year 2	Year 3
2017	1399.68			$1399.68 \times 0.1599 = 224$
2018	1542.96		$1542.96 \times 0.0378 = 58$	$1542.96 \times 0.1599 = 247$
2019	1508.40	$1508.40 \times 0.2777 = 419$	$1508.40 \times 0.0378 = 57$	$1508.40 \times 0.1599 = 241$

The total reserve needed for payments in 2020 is therefore $224 + 58 + 419 = 701$.