ACSC/STAT 4703, Actuarial Models II
Fall 2020
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Homework Sheet 4
Model Solutions

Basic Questions

1. *An insurance company sells insurance. It estimates that the standard deviation of the aggregate annual claim is \$4,521 and the mean is \$1,020.*

   (a) How many years history are needed for an individual or group to be assigned full credibility? (Use $r = 0.05$, $p = 0.90$.)

   The variance is and individual's mean aggregate claim over n years is $\frac{4521^2}{n}$. For an individual with average claims, the coefficient of variation is $\frac{4521}{1020\sqrt{n}}$. We want the relative error to have absolute value less than 0.05 with probability 0.90. That is, we want

$$\Phi\left(\frac{0.05 \times 1020\sqrt{n}}{4521}\right) = 0.95$$
$$\frac{0.05 \times 1020\sqrt{n}}{4521} = 1.644854$$
$$n = \left(\frac{1.644854 \times 4521}{0.05 \times 1020}\right)^2$$
$$= 21260.9845777$$

   so 21,261 years are needed for full credibility.

   The standard premium for this policy is \$1,020. A company has claimed a total of \$8,072 in the last 23 years.

   (b) What is the Credibility premium for this company, using limited fluctuation credibility?

   Using limited fluctuation credibility, the credibility of this company's experience is $Z = \sqrt{\frac{23}{21260.9845777}} = 0.0328906329971$. Thus, the credibility premium is $0.0328906329971 \times \frac{8072}{23} + 0.967109367003 \times 1020 = \$997.99$.

2. *A home insurance company classifies houses as high, medium or low risk. Annual claims from high risk houses follow a Gamma distribution with $\alpha = 4$ and $\theta = 5000$. Annual claims from medium risk houses follow a Gamma distribution with $\alpha = 8$ and $\theta = 1400$. Annual claims from low risk houses follow a Gamma with $\alpha = 14$ and $\theta = 600$. 15% of houses are high risk, 65% are medium risk and 20% are low risk.*

(a) Calculate the expectation and variance of the aggregate annual claims from a randomly chosen house.

Risk	Probability	Expected Claims	Variance of claims
High	0.15	20,000	100,000,000
Medium	0.65	11,200	15,680,000
Low	0.20	8,400	5,040,000

Thus the overall expected claim is $20000\times0.15+11200\times0.65+8400\times0.2 = \$11,960$ and the variance is

$$\begin{aligned}&100000000\times0.15+15680000\times0.65+5040000\times0.20\\&\quad+0.15(20000-11960)^2+0.65(11200-11960)^2+0.20(8400-11960)^2\\&\quad=38806400\end{aligned}$$

(b) Given that a homeowner's annual claims over the past 4 years are \$4,000, \$250 and \$1,100, what are the expectation and variance of the homeowners' claims next year?

The likelihood of these claims for a high-risk home is

$$\frac{4000^3e^{-\frac{4000}{5000}}250^3e^{-\frac{250}{5000}}1100^3e^{-\frac{1100}{5000}}}{5000^{12}\Gamma(4)^3}=8.65743334751\times10^{-21}$$

For a medium risk home, the likelihood is

$$\frac{4000^7e^{-\frac{4000}{1400}}250^7e^{-\frac{250}{1400}}1100^7e^{-\frac{1100}{1400}}}{1400^{24}\Gamma(8)^3}=1.03695280072\times10^{-25}$$

For a low risk home, the likelihood is

$$\frac{4000^{13}e^{-\frac{4000}{600}}250^{13}e^{-\frac{250}{600}}1100^{13}e^{-\frac{1100}{600}}}{600^{42}\Gamma(14)^3}=3.98522009444\times10^{-33}$$

The overall likelihood of these claims is therefore

$$3.985220\times10^{-33}\times0.2+1.036953\times10^{-25}\times0.65+8.657433\times10^{-21}\times0.15=1.29868240406\times10^{-21}$$

Therefore, the posterior probabilities of the three classes are

$$\frac{8.65743334751\times10^{-21}\times0.15}{1.29868240406\times10^{-21}}=0.999948099759$$
$$\frac{1.03695280072\times10^{-25}\times0.65}{1.29868240406\times10^{-21}}=5.19002427661\times10^{-5}$$
$$\frac{3.98522009444\times10^{-33}\times0.2}{1.29868240406\times10^{-21}}=6.13732823665\times10^{-13}$$

Thus the posterior mean and variance are

$20000\times0.999948099759+11200\times5.19002427661\times10^{-5}+8400\times6.13732823665\times 10^{-13}=19999.5432779$ and the variance is

$$100000000\times0.999948+15680000\times5.190024\times10^{-5}+5040000\times6.137328\times10^{-13}$$
$$+0.9999(20000-19999.5)^2+5.1900\times10^{-5}(11200-19999.5)^2+6.1373\times10^{-13}(8400-19999.5)^2$$
$$=99999642.718$$

Standard Questions

3. *For a certain insurance policy, the book premium is based on average claim frequency of 4.9 claims per year, and average claim severity of \$4,200. The standard for full credibility is 50 policy years for claim frequency and 230 claims for severity. The insurance company wants to change the standard for full credibility to a single standard (in terms of policy years) for aggregate claims. A particular group has 100 claims for a total of \$282,000, in 27 policy years of history. The insurance company wants the new standard to give the same premium for this group. What should the new standard be?*

   In the current standard, the credibility for claim frequency is $Z=\sqrt{\frac{27}{50}}=0.734846922835$ and for severity is $\sqrt{\frac{100}{230}}=0.659380473396$. Therefore the credibility estimate for frequency is $0.734846922835\times\frac{100}{27}+0.265153077165\times 4.9=4.02090534787$ and for severity is $0.659380473396\times2820+0.340619526604\times 4200=3290.05494672$. Therefore the current credibility premium is $4.02090534787\times3290.05494672=13228.9995301$. For credibility $Z=\sqrt{\frac{27}{n_0}}$, the premium is $\frac{282000}{27}Z+4.9\times4200(1-Z)$. Setting this equal to

13228.9995301 gives

$$\begin{aligned}
\frac{282000}{27}Z + 4.9 \times 4200(1-Z) &= 13228.9995301\\
20580 - 10135.5555556Z &= 13228.9995301\\
Z &= \frac{7351.0004699}{10135.5555556}\\
&= 0.72526862781\\
\sqrt{\frac{27}{n_0}} &= 0.72526862781\\
\frac{27}{n_0} &= 0.526014582485\\
n_0 &= \frac{27}{0.526014582485}\\
&= 51.3293754566
\end{aligned}$$

So they should set the new standard for full credibility to 51.3293754566 policy years in order to get the same premium.

4. *Aggregate claims for an individual are believed to follow a gamma distribution with $\alpha = 0.8$ and Θ varying between individuals. For a randomly chosen individual, Θ follows an inverse gamma distribution with $\alpha = 3$ and $\theta = 2000$. The insurance company uses limited fluctuation credibility with $r = 0.05$ and $p = 0.95$ to determine an individual's premium. If an individual has 6 years of past history, for what value of total claims during these 6 years would the limited fluctuation credibility premium equal the fair premium (using the Bayesian method)?*

   For an individual with claims $x_1, \ldots, x_6$, the likelihood of these claims is

$$\frac{(x_1 \cdots x_6)^{\alpha-1} e^{-\frac{x_1+\cdots+x_6}{\theta}}}{\theta^{6\alpha}\Gamma(\alpha)^6} = \left(\frac{(x_1 \cdots x_6)^{\alpha-1}}{\Gamma(\alpha)^6}\right)\theta^{-6\alpha}e^{-\frac{x_1+\cdots+x_6}{\theta}}$$

   The posterior distribution of θ is therefore given by

$$f_{\Theta|X}(\theta) = \frac{\theta^{-4}e^{-\frac{2000}{\theta}}\theta^{-4.8}e^{-\frac{x_1+\cdots+x_6}{\theta}}}{\int_0^\infty \theta^{-8.8}e^{-\frac{2000}{\theta}}e^{-\frac{x_1+\cdots+x_6}{\theta}}\,d\theta}$$

   This is an inverse gamma distribution with $\alpha = 7.8$ and $\theta = 2000 + x_1 + \cdots + x_6$. The marginal expected aggregate claims is $0.8\mathbb{E}(\Theta)$. For this inverse gamma distribution, we have $\mathbb{E}(\Theta) = \frac{2000+x_1+\cdots+x_6}{7.8-1}$. Therefore the fair premium is $\frac{0.8}{6.8}(2000 + x_1 + \cdots + x_6)$.

   For a particular individual, the coefficient of variation is

$$\frac{\sqrt{\alpha\Theta^2}}{\alpha\Theta} = \alpha^{-\frac{1}{2}} = 0.8^{-\frac{1}{2}} = 1.11803398875$$

Therefore, the standard for full credibility is obtained by solving

$$\begin{aligned}\frac{0.05\sqrt{n_0}}{1.11803398875} &= 1.959964\\ n_0 &= \frac{1.959964^2}{0.05^2\times 0.8}\\ &= 1920.72944065\end{aligned}$$

The book premium is $0.8\times\frac{2000}{3-1}=800$. Thus the credibility premium is $\sqrt{\frac{6}{1920.72944065}}\frac{x_1+\cdots+x_6}{6}+\left(1-\sqrt{\frac{6}{1920.72944065}}\right)800$. Therefore, in order for the credibility premium to equal the fair premium, we must have

$$\begin{aligned}\sqrt{\frac{6}{1920.72944065}}\frac{x_1+\cdots+x_6}{6}+\left(1-\sqrt{\frac{6}{1920.72944065}}\right)800 &= \frac{0.8}{6.8}(2000+x_1+\cdots+x_6)\\ 0.108331878247(x_1+\cdots+x_6) &= 519.993015586\\ x_1+\cdots+x_6 &= 4800\end{aligned}$$

5. *An insurance company has 4 years of past history on a marine insurance policy, denoted X_1, X_2, X_3, X_4. It uses a formula $\hat{X}_5=\alpha_0+\alpha_1X_1+\alpha_2X_2+\alpha_3X_3+\alpha_4X_4$ to calculate the credibility premium in the fifth year. It has the following information on the policy:*

   - *In Year 1, the expected aggregate claim was $32,000.*
   - *Expected aggregate claims increase by 4% per year.*
   - *The coefficient of variation of the aggregate claims is 0.8 in every year.*
   - *The correlation (recall Corr$(X,Y)=\frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}}$) between aggregate claims in years i and j is $e^{-\frac{|i-j|}{2}}$ for all $i\neq j$.*

   Find a set of equations which can determine the values of α_0, α_1, α_2, α_3 and α_4. [You do not need to solve these equations.]

   We have $\mathbb{E}(X_i)=32000(1.04)^{i-1}$ and $\mathrm{Var}(X_i)=0.8^2\times32000^2(1.04)^{2(i-1)}=655360000(1.04)^{2(i-1)}$. Finally, for $i\neq j$ we get

$$\begin{aligned}\mathrm{Cov}(X_i,X_j) &= e^{-\frac{|i-j|}{2}}\sqrt{655360000(1.04)^{2(i-1)}\times 655360000(1.04)^{2(j-1)}}\\ &= e^{-\frac{|i-j|}{2}}655360000(1.04)^{i+j-2}\end{aligned}$$

   Recall that the greatest accuracy credibility equations give us

$$\begin{aligned}\mathbb{E}(X_5) &= \alpha_0+\alpha_1\mathbb{E}(X_1)+\alpha_2\mathbb{E}(X_2)+\alpha_3\mathbb{E}(X_3)+\alpha_4\mathbb{E}(X_4)\\ \mathrm{Cov}(X_5,X_i) &= \alpha_1\,\mathrm{Cov}(X_1,X_i)+\alpha_2\,\mathrm{Cov}(X_2,X_i)+\alpha_3\,\mathrm{Cov}(X_3,X_i)+\alpha_4\,\mathrm{Cov}(X_4,X_i)\end{aligned}$$

Substituting the values we have gives

$$
\begin{aligned}
32000(1.04)^4 &= \alpha_0 + 32000\alpha_1 + 32000(1.04)^1\alpha_2 + 32000(1.04)^2\alpha_3 + 32000(1.04)^3\alpha_4 \\
e^{-2}25600^2(1.04)^4 &= 25600^2\alpha_1 + e^{-0.5}25600^2(1.04)\alpha_2 + e^{-1}25600^2(1.04)^2\alpha_3 + e^{-1.5}25600^2(1.04)^3\alpha_4 \\
e^{-1.5}25600^2(1.04)^5 &= e^{-0.5}25600^2(1.04)\alpha_1 + 25600^2(1.04)^2\alpha_2 + e^{-0.5}25600^2(1.04)^3\alpha_3 + e^{-1}25600^2(1.04)^4\alpha_4 \\
e^{-1}25600^2(1.04)^6 &= e^{-1}25600^2(1.04)^2\alpha_1 + e^{-0.5}25600^2(1.04)^3\alpha_2 + 25600^2(1.04)^4\alpha_3 + e^{-0.5}25600^2(1.04)^5\alpha_4 \\
e^{-0.5}25600^2(1.04)^7 &= e^{-1.5}25600^2(1.04)^3\alpha_1 + e^{-1}25600^2(1.04)^4\alpha_2 + e^{-0.5}25600^2(1.04)^5\alpha_3 + 25600^2(1.04)^6\alpha_4
\end{aligned}
$$

[We can simplify and solve these equations reasonably straightforwardly:

$$
\begin{aligned}
1 &= \frac{\alpha_0}{32000(1.04)^4} + (1.04)^{-4}\alpha_1 + (1.04)^{-3}\alpha_2 + (1.04)^{-2}\alpha_3 + (1.04)^{-1}\alpha_4 \\
1 &= e^2(1.04)^{-4}\alpha_1 + e^{1.5}(1.04)^{-3}\alpha_2 + e^1(1.04)^{-2}\alpha_3 + e^{0.5}(1.04)^{-1}\alpha_4 \\
1 &= e^1(1.04)^{-4}\alpha_1 + e^{1.5}(1.04)^{-3}\alpha_2 + e^1(1.04)^{-2}\alpha_3 + e^{0.5}(1.04)^{-1}\alpha_4 \\
1 &= (1.04)^{-4}\alpha_1 + e^{0.5}(1.04)^{-3}\alpha_2 + e^1(1.04)^{-2}\alpha_3 + e^{0.5}(1.04)^{-1}\alpha_4 \\
1 &= e^{-1}(1.04)^{-4}\alpha_1 + e^{-0.5}(1.04)^{-3}\alpha_2 + (1.04)^{-2}\alpha_3 + e^{0.5}(1.04)^{-1}\alpha_4
\end{aligned}
$$

Letting $\beta_i = \alpha_i(1.04)^{i-5}$ and $\gamma_i = \beta_i e^{\frac{5-i}{2}}$ these become

$$
\begin{aligned}
1 &= \frac{\alpha_0(1.04)^{-4}}{32000} + \beta_1 + \beta_2 + \beta_3 + \beta_4 \\
1 &= \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 \\
1 &= e^{-1}\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 \\
1 &= e^{-2}\gamma_1 + e^{-1}\gamma_2 + \gamma_3 + \gamma_4 \\
1 &= e^{-3}\gamma_1 + e^{-2}\gamma_2 + e^{-1}\gamma_3 + \gamma_4
\end{aligned}
$$

which is easy to solve as $\gamma_1 = \gamma_2 = \gamma_3 = 0$, $\gamma_4 = 1$, so $\alpha_1 = \alpha_2 = \alpha_3 = 0$, $\alpha_4 = e^{-0.5}(1.04)$, and $\alpha_0 = 32000(1 - e^{-0.5})(1.04)^5$.
]