

MATH/STAT 4703, Actuarial
Models II
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Formula Sheet

General Mathematics

- Quadratic Formula: Solution to $ax^2 + bx + c = 0$ is $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
- Gamma function: $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx$ satisfies $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$.

Moments

Centralised moments in terms of uncentralised moments:

$$\begin{aligned}\mu_2 &= \mu'_2 - \mu^2 \\ \mu_3 &= \mu'_3 - 3\mu\mu'_2 + 2\mu^3 \\ \mu_4 &= \mu'_4 - 4\mu\mu'_3 + 6\mu^2\mu'_2 - 3\mu^4\end{aligned}$$

Risk Measures

- Standard deviation principle $r = \mu + a\sigma$.
- Value at Risk $r = \pi_p$.

- Tail Value at Risk $r = \frac{\int_{\pi_p}^\infty xf(x)dx}{1-p}$
 $= \pi_p + \frac{\int_{\pi_p}^\infty S(x)dx}{1-p}$

Continuous Distributions: Transformed Beta family

Transformed Beta

Inverse of Transformed Beta with $\alpha = \tau$, $\tau = \alpha$, $\theta = \frac{1}{\theta}$.

Density function	$f(x) = \left(\frac{\Gamma(\alpha+\tau)}{\Gamma(\alpha)\Gamma(\tau)}\right) \frac{\gamma\left(\frac{x}{\theta}\right)^{\tau\gamma}}{x\left(1+\left(\frac{x}{\theta}\right)^\gamma\right)^{\alpha+\tau}}$
Mean	$\theta \frac{\Gamma(\tau+\frac{1}{\gamma})\Gamma(\alpha-\frac{1}{\gamma})}{\Gamma(\tau)\Gamma(\alpha)}$
Raw Moments	$\mu'_k = \theta^k \frac{\Gamma(\tau+\frac{k}{\gamma})\Gamma(\alpha-\frac{k}{\gamma})}{\Gamma(\tau)\Gamma(\alpha)}$
Moment Generating Function	Undefined

Burr

Transformed Beta with $\tau = 1$.

Density function	$f(x) = \frac{\alpha\gamma\left(\frac{x}{\theta}\right)^\gamma}{x\left(1+\left(\frac{x}{\theta}\right)^\gamma\right)^{\alpha+1}}$
Survival Function	$\frac{1}{\left(1+\left(\frac{x}{\theta}\right)^\gamma\right)^\alpha}$
Mean	$\theta \frac{\Gamma(\alpha-\frac{1}{\gamma})\Gamma(\frac{1}{\gamma})}{\Gamma(\alpha)}$
Raw Moments	$\mu'_n = \theta^n \frac{n\Gamma(\alpha-\frac{n}{\gamma})\Gamma(\frac{n}{\gamma})}{\Gamma(\alpha)}$
Moment Generating Function	Undefined

Inverse Burr

Transformed Beta with $\alpha = 1$.

Density function	$f(x) = \frac{\tau\gamma\left(\frac{x}{\theta}\right)^{\gamma\tau}}{x\left(1+\left(\frac{x}{\theta}\right)^\gamma\right)^{\tau+1}}$
Survival Function	$\frac{1}{\left(1+\left(\frac{x}{\theta}\right)^\gamma\right)^\alpha}$
Mean	$\theta \frac{\Gamma(\tau+\frac{1}{\gamma})\Gamma(1-\frac{1}{\gamma})}{\Gamma(\tau)}$
Raw Moments	$\mu'_k = \theta^k \frac{\Gamma(\tau+\frac{k}{\gamma})\Gamma(1-\frac{k}{\gamma})}{\Gamma(\tau)}$
Moment Generating Function	Undefined

Generalised Pareto

Transformed Beta with $\gamma = 1$.

Density function	$f(x) = \left(\frac{\Gamma(\alpha+\tau)}{\Gamma(\alpha)\Gamma(\tau)} \right) \frac{\left(\frac{x}{\theta}\right)^\tau}{x(1+\left(\frac{x}{\theta}\right))^{\alpha+\tau}}$
Mean	$\theta \frac{\tau}{\alpha-1}$
Raw Moments	$\mu'_k = \theta^k \frac{\Gamma(\tau+k)\Gamma(\alpha-k)}{\Gamma(\tau)\Gamma(\alpha)}$
Moment Generating Function	Undefined

Density function	$f(x) = \frac{\gamma\left(\frac{x}{\theta}\right)^\gamma}{x(1+\left(\frac{x}{\theta}\right)^\gamma)^{\gamma+1}}$
Survival function	$S(x) = \frac{1}{(1+\left(\frac{x}{\theta}\right)^\gamma)^\gamma}$
Mean	$\theta \frac{\Gamma(\gamma-\frac{1}{\gamma})\Gamma(\frac{1}{\gamma})}{\Gamma(\gamma)}$
Raw Moments	$\mu'_k = \theta^k \frac{\Gamma(1+\frac{k}{\gamma})\Gamma(\gamma-\frac{k}{\gamma})}{\Gamma(\gamma)}$
Variance	
Moment Generating Function	Undefined

Pareto

Transformed Beta with $\tau = \gamma = 1$.

Density function	$f(x) = \frac{\alpha}{\theta(1+\left(\frac{x}{\theta}\right))^{\alpha+1}}$
Survival Function	$\frac{1}{(1+\left(\frac{x}{\theta}\right))^\alpha}$
Mean	$\frac{\theta}{\alpha-1}$ (if $\alpha > 1$)
Variance	$\frac{\theta^2\alpha}{(\alpha-1)^2(\alpha-2)}$ (if $\alpha > 2$)
Raw Moments	$\mu'_k = \theta^k \frac{\Gamma(1+k)\Gamma(\alpha-k)}{\Gamma(\alpha)}$
Moment Generating Function	Undefined

Inverse Pareto

Transformed Beta with $\alpha = \gamma = 1$.

Density function	$f(x) = \frac{\tau\left(\frac{\theta}{x}\right)}{x(1+\left(\frac{\theta}{x}\right))^{\tau+1}}$
Survival Function	$1 - \frac{1}{(1+\left(\frac{\theta}{x}\right))^\tau}$
Mean	undefined
Moment Generating Function	Undefined

log-logistic

Transformed Beta with $\alpha = \tau = 1$.

Density function	$f(x) = \frac{\gamma\left(\frac{x}{\theta}\right)^\gamma}{x(1+\left(\frac{x}{\theta}\right)^\gamma)^2}$
Survival Function	$\frac{1}{(1+\left(\frac{x}{\theta}\right)^\gamma)}$
Mean	$\theta\Gamma\left(1+\frac{1}{\gamma}\right)\Gamma\left(1-\frac{1}{\gamma}\right)$
Raw Moments	$\mu'_k = \theta^k\Gamma\left(1+\frac{k}{\gamma}\right)\Gamma\left(1-\frac{k}{\gamma}\right)$
Moment Generating Function	Undefined

Paralogistic

Transformed Beta with $\tau = 1, \alpha = \gamma$.

Inverse Paralogistic

Transformed Beta with $\alpha = 1$, $\tau = \gamma$.

Density function $f(x) = \frac{\gamma(\frac{\theta}{x})^\gamma}{x(1+(\frac{\theta}{x})^\gamma)^{\gamma+1}}$
 Survival function $S(x) = 1 - \frac{1}{(1+(\frac{\theta}{x})^\gamma)^\gamma}$
 Mean

Raw Moments $\mu'_k = \theta^k \frac{\Gamma(\gamma + \frac{k}{\gamma})\Gamma(1 - \frac{k}{\gamma})}{\Gamma(\gamma)}$

Variance

Excess loss

Moment Generating Function Undefined

Continuous Distributions: Transformed Gamma family

Transformed Gamma

Limit of Transformed Beta as $\alpha \rightarrow \infty$ and $\theta \rightarrow \infty$ with $\alpha\theta^\alpha = \xi$.

Density function $f(x) = \frac{\tau(\frac{\xi}{x})^{\tau\alpha} e^{-(\frac{\xi}{x})^\tau}}{x\Gamma(\alpha)}$

Mean $\mu = \theta \frac{\tau(\alpha + \frac{1}{\tau})}{\tau(\alpha)}$

Raw moments $\mu'_n = \theta^n \frac{\Gamma(\alpha + \frac{n}{\tau})}{\Gamma(\alpha)}$

Gamma

Transformed Gamma with $\tau = 1$

Density function $f(x) = \frac{(\frac{x}{\theta})^\alpha e^{-(\frac{x}{\theta})}}{x\Gamma(\alpha)}$

Survival function $S(x) = e^{-\frac{x}{\theta}} (1 + \dots + \frac{(\frac{x}{\theta})^{\alpha-1}}{(\alpha-1)!})$
 (for $\alpha \in \mathbb{Z}^+$)

Mean $\mu = \theta\alpha$

Raw moments $\mu'_n = \theta^n \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}$

Variance $\mu_n = \theta^n \alpha$

Moment Generating Function $M(t) = \frac{1}{(1-\theta t)^\alpha}$

Function

Weibull

Transformed Gamma with $\alpha = 1$

Density function $f(x) = \frac{\tau(\frac{x}{\theta})^{\tau-1} e^{-(\frac{x}{\theta})^\tau}}{x}$

Survival function $e^{-(\frac{x}{\theta})^\tau}$

Mean $\mu = \theta\Gamma(1 + \frac{1}{\tau})$

Raw moments $\mu'_n = \theta^n \Gamma(1 + \frac{n}{\tau})$

Exponential

Transformed Gamma with $\alpha = \tau = 1$

Density function $f(x) = \frac{e^{-(\frac{x}{\theta})}}{\theta}$

Survival function $e^{-\frac{x}{\theta}}$

Mean $\mu = \theta$

Raw moments $\mu'_n = n!\theta^n$

Variance $\mu_n = \theta^n$

Excess loss $\theta e^{-\frac{x}{\theta}}$

Moment Generating Function $M(t) = \frac{1}{1-\theta t}$

Function

Inverse Transformed Gamma

Inverse of transformed gamma with $\theta = \frac{1}{\theta}$.

Density function $f(x) = \frac{\tau(\frac{\theta}{x})^{\tau\alpha} e^{-(\frac{\theta}{x})^\tau}}{x\Gamma(\alpha)}$

Mean $\mu = \theta \frac{\Gamma(\alpha + \frac{1}{\tau})}{\Gamma(\alpha)}$ (if $\tau\alpha > 1$)

Raw moments $\mu'_n = \theta^n \frac{\Gamma(\alpha + \frac{n}{\tau})}{\Gamma(\alpha)}$ (if $\tau\alpha > n$)

Inverse Gamma

Inverse Transformed Gamma with $\tau = 1$. Inverse of gamma distribution with $\theta = \frac{1}{\theta}$.

Density function $f(x) = \frac{(\frac{\theta}{x})^\alpha e^{-(\frac{\theta}{x})}}{x\Gamma(\alpha)}$

Survival function $S(x) = 1 - e^{-\frac{\theta}{x}} (1 + \dots + \frac{(\frac{\theta}{x})^{\alpha-1}}{(\alpha-1)!})$
 (for $\alpha \in \mathbb{Z}^+$)

Mean $\mu = \frac{\theta}{\alpha-1}$ (if $\alpha > 1$)

Raw moments $\mu'_n = \theta^n \frac{\Gamma(\alpha-n)}{\Gamma(\alpha)}$ (if $\alpha > n$)

Variance $\mu_2 = \frac{\theta^2}{(\alpha-1)^2(\alpha-2)}$

Inverse Weibull

Inverse Transformed Gamma with $\alpha = 1$. Inverse of Weibull distribution with $\theta = \frac{1}{\theta}$.

Density function	$f(x) = \frac{\tau(\frac{\theta}{x})^\tau e^{-\left(\frac{\theta}{x}\right)^\tau}}{x}$
Survival function	$1 - e^{-\left(\frac{\theta}{x}\right)^\tau}$
Mean	$\mu = \theta\Gamma\left(1 - \frac{1}{\tau}\right)$ (if $\tau > 1$)
Raw moments	$\mu'_n = \theta^n\Gamma\left(1 - \frac{n}{\tau}\right)$ (if $\tau > n$)
Moment Generating Function	Undefined

Density function	$f(x) = \frac{1}{\sqrt{2\pi}\sigma x} e^{-\frac{(\log(x)-\mu)^2}{2\sigma^2}}$
Mean	$e^{\mu + \frac{\sigma^2}{2}}$
Variance	$e^{2\mu + \sigma^2} (e^{\sigma^2} - 1)$
Moment Generating Function	undefined

Inverse Exponential

Inverse Transformed Gamma with $\tau = \alpha = 1$, inverse of exponential with $\theta = \frac{1}{\theta}$.

Density function	$f(x) = \frac{\theta e^{-\left(\frac{\theta}{x}\right)}}{x^2}$
Survival function	$1 - e^{-\frac{\theta}{x}}$
Mean	Undefined

Linear Exponential Family

Density	$f_\theta(x) = \frac{p(x)e^{r(\theta)x}}{q(\theta)}$
mean	$\mu(\theta) = \frac{q'(\theta)}{q(\theta)r'(\theta)}$
Variance	$\mu_2(\theta) = \frac{\mu'(\theta)}{r'(\theta)}$

Normal

Density function	$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$
Mean	$\mu = \mu$
Variance	σ^2
Moment Generating Function	$M(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$

Uniform

Density function	$f(x) = \frac{1}{b-a}$ (for $a < x < b$)
Survival function	$S(x) = \frac{b-x}{b-a}$ (for $a \leq x \leq b$)
Mean	$\mu = \frac{a+b}{2}$
Variance	$\frac{(b-a)^2}{12}$
Moment Generating Function	$M(t) = \frac{e^{bt} - e^{at}}{(b-a)t}$

Log-Normal

Exponential of a normal distribution.

Discrete Distributions

Binomial

Probability	$p_k = \binom{n}{k} p^k (1-p)^{n-k}$
mean	$\mu = np$
raw moments	$\mathbb{E}(X \cdots (X+1-m)) = n \cdots (n+1-m)p^m$
Variance	$\mu_2 = np(1-p)$
p.g.f.	$P(z) = (1-p+pz)^n$
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$(a, b, 0)$ -class	$a = -\frac{p}{1-p}, b = \frac{(n-1)p}{1-p}$
zero-truncated probability	$p_1^T = \frac{np(1-p)^{n-1}}{1-(1-p)^n}$

Poisson

Limit of binomial as $n \rightarrow \infty, p \rightarrow 0$ with $np = \lambda$.

Probability	$p_k = e^{-\lambda} \frac{\lambda^k}{k!}$
mean	$\mu = \lambda$
raw moments	$\mathbb{E}(X(X-1) \cdots (X+1-m)) = \lambda^m$
Variance	$\mu_k = \lambda$
p.g.f.	$P(z) = e^{\lambda(z-1)}$
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$(a, b, 0)$ -class	$a = 0, b = \lambda$
zero-truncated probability	$p_1^T = \frac{\lambda e^{-\lambda}}{1-e^{-\lambda}}$

Negative Binomial

- Gamma mixture of Poisson distributions where λ follows a gamma distribution with $\theta = \beta$ and $\alpha = r$.
- Number of successes before r failures if probability of success is $\frac{\beta}{1+\beta}$.
- Compound Poisson-Logarithmic distribution, where $\lambda = r \log\left(\frac{1}{1+\beta}\right)$ and $a = \frac{\beta}{1+\beta}$.

Probability	$p_k = \binom{k+r-1}{k} \left(\frac{\beta}{1+\beta}\right)^k \left(\frac{1}{1+\beta}\right)^r$ $= \frac{r(r+1)\cdots(r+k-1)}{k!} \left(\frac{\beta}{1+\beta}\right)^k \left(\frac{1}{1+\beta}\right)^r$
mean	$\mu = r\beta$
Variance	$\mu_n = r\beta(1+\beta)\cdots(n-1+\beta)$
p.g.f.	$P(z) = \left(\frac{1}{1+\beta-\beta z}\right)^r$
$(a, b, 0)$ -class	$a = \frac{\beta}{1+\beta}, b = \frac{(r-1)\beta}{1+\beta}$
zero-truncated probability	$p_1^T = \frac{r\beta}{(1+\beta)^{r+1} - (1+\beta)}$

$(a, b, 0)$ and $(a, b, 1)$ Classes

$p_k = \left(a + \frac{b}{k}\right) p_{k-1}$ for $k > 1$ (and for $k > 0$ in the $(a, b, 0)$ class).

mean	$\mu = \frac{a+b}{1-a}$
Variance	$\mu_2 = \frac{a+b}{(1-a)^2}$
p.g.f.	$P(z) = \left(\frac{1-az}{1-a}\right)^{-\left(1+\frac{b}{a}\right)}$
zero-truncated mean	$\mu = \frac{a+b}{(1-a)\left(1-(a+b)^{1+\frac{b}{a}}\right)}$
zero-truncated probability	$p_1^T = \frac{a+b}{(1-a)^{-\left(1+\frac{b}{a}\right)} - 1}$

Logarithmic distribution

Negative binomial with $r = 0$. $(a, b, 1)$ -class with $a + b = 0$, $a = \frac{\beta}{1+\beta}$.

zero-truncated probability	$p_1^T = \frac{-a}{\log(1-a)} = \frac{\beta}{(1+\beta)\log(1+\beta)}$
probability	$p_n = \frac{a^{n-1}}{n} p_1$
mean	$\mu = \frac{-a}{(1-a)\log(1-a)} = \frac{\beta}{\log(1+\beta)}$
Variance	$\mu_2 = \frac{p_1 - p_1^2}{(1-a)^2} = \frac{\beta(1+\beta)}{\log(1+\beta)} - \frac{\beta}{\log(1+\beta)^2}$
p.g.f.	$P(z) = -\frac{\log(1-az)}{\log(1-a)}$
$(a, b, 1)$ -class	$a = \frac{\beta}{1+\beta}, b = -\frac{\beta}{1+\beta}$

Compound Distributions

Moments:

Let the moments of the primary distribution be μ, μ_2, μ_3, \dots , and the moments of the secondary distribution by ν, ν_2, ν_3, \dots . The moments of the compound distribution are given by:

$$\begin{aligned} \mu\nu \\ \mu\nu_2 + \mu_2\nu^2 \\ \mu\nu_3 + \mu_2\nu\nu_2 + \mu_3\nu^3. \end{aligned}$$

Recursive formula:

If the primary distribution is a member of the $(a, b, 1)$ -class, the probability mass function is defined as

$$f_S(k) = \frac{(p_1 - (a+b)p_0)f_X(k) + \sum_{i=1}^k \left(a + \frac{bi}{k}\right) f_X(i)f_S(k-i)}{1 - af_X(0)}$$

where:

- f_X is the probability mass function of the secondary distribution
- f_S is the probability mass function of the compound distribution
- p_n is the probability that the primary distribution is n (so $p_n = \left(a + \frac{b}{n}\right) p_{n-1}$)

Non-parametric Estimators

Greenwood's formula

$$\text{Var}(S_n(y_j)) \approx S_n(y_j)^2 \sum_{i=1}^j \frac{s_i}{r_i(r_i - s_i)}$$

where

- y_i is the i th observed data point in increasing order.
- s_i is the frequency of the observation y_i
- r_i is the size of the risk set at observation y_i .

- Unique observed values are $t = y_0 < y_1 < \dots < y_k < y_{k+1} = u$
- t is the (left) truncation point (can be $-\infty$ or 0 if no truncation).
- u is the (right) censorship point (can be ∞ if no censorship).

Log-transformed Confidence intervals

$[S_n(x)^{\frac{1}{\sigma}}, S_n(X)^U]$, where $U = e^{\Phi^{-1}(\frac{\alpha}{2}) \frac{\sigma}{S_n(x) \log(S_n(x))}}$.

- α is the confidence level (so for a 95% confidence interval, $\alpha = 0.05$).
- σ is the standard deviation of the estimator $S_n(x)$.

Information Criteria

- Akaike information criterion (AIC) $l(\theta; x) - 2p$
- Schwartz Bayesian criterion/Bayes information criterion (BIC) $l(\theta; x) - \frac{p \log(n)}{2}$

Hypothesis Tests

Anderson-Darling test

- Test statistic $n \int_t^u \frac{(F_n(x) - F^*(x))^2}{F^*(x)(1 - F^*(x))} f^*(x) dx$
- For complete data, given by the formula:

$$-nF^*(u) + n \sum_{i=1}^k (F_n(y_i))^2 (\log(F^*(y_{i+1})) - \log(F^*(y_i)))$$

$$+ n \sum_{i=0}^k (1 - F_n(y_i))^2 (\log(1 - F^*(y_i)) - \log(1 - F^*(y_{i+1})))$$

where

- n is sample size.