2 Let the moments of the primary distribution be μ_1, μ_2, μ_3 (and similar notation for raw moments). Let the moments of the secondary distribution be ν_1, ν_2, ν_3 (and similar notation for raw moments).

Recall that $P(z) = M(log z)$, so $P'(z) = \frac{M'(log(z))}{z}$, $P''(z) = \frac{M''(log(z))-M'(log(z))}{z^2}$, and $P'''(z) = \frac{M'''(log(z))-3M''(log(z))+2M'(log(z))}{z^3}$. In particular, $P'(1) = \mu$, $P''(1) = \mu_2' - \mu$ and $P'''(1) = \mu_3' - 3\mu_2' + 2\mu$.

m.g.f. of compound model is $P(M(z))$ first 3 derivatives of this at 0 are:

$$M'(0)P'(M(0)) = M'(0)P'(1) = \mu\nu$$
$$M''(0)P'(1) + M'(0)^2P''(1) = \mu\nu_2' + (\mu_2' - \mu)\nu^2$$
$$M'''(0)P'(1) + 3M''(0)M'(0)P''(1) + M'(0)^3P'''(1) = \mu\nu_3' + 3(\mu_2' - \mu)\nu\nu_2' + (\mu_3' - 3\mu_2' + 2\mu)\nu^3$$

3

For a given claim, the amount reimbursed has mean $1000 + 0.8 \times 500 = 1400$, and variance $500^2 + 0.8^2 \times 300^2 + 2 \times 0.8 \times 100000 = 467,600$.

The mean of the aggregate claims is therefore: $4 \times 1400 = 5600$. The variance is given by the law of total variance

$$\begin{aligned} \mathrm{Var}(A) &= \mathbb{E}(N \, \mathrm{Var}(X_i)) + \mathrm{Var}(N\mathbb{E}(X_i)) \\ &= \mathbb{E}(N) \, \mathrm{Var}(X_i) + \mathbb{E}(X_i)^2 \, \mathrm{Var}(N) \\ &= 4 \times 467600 + 1400^2 \times 4 \\ &= 9710400 \end{aligned}$$

Alternatively, the raw second moment is

$$4 \times (467600 + 1400^2) + (20 - 4) \times 1400^2 = 1,870,400 + 7,840,000 + 31,360,000 = 41,070,400$$

The variance is this minus 5600^2, which is $9,710,400$.

The standard deviation is the square root of this or $3,116.15$.

4

We have

n	$P(N=n)$	Z	$P(A>130\|N=n)$
0	0.4	∞	0
1	0.3	0.8571	0.1957
2	0.2	-1.414	0.9214
3	0.1	-2.804	0.9975

So the probability is $0+0.3\times 0.1957+0.2\times 0.9214+0.1\times 0.9975=0.0587+0.1843+0.0997=0.3427$.

5

mean=$4 \times 6 \times 16 = 384$. Variance=$\mu\nu_2 + \mu_2\nu^2 = 6 \times 16 \times \frac{8^2}{12} + 4^2 \times 16 \times 6 \times 7 = 512 + 512 \times 21$ The standard deviation is therefore $32\sqrt{11}$. 95th percentile is 1.645 standard deviations above the mean or $384 + 52.64\sqrt{11} = 558.59$.

6

Prob of stop-loss is $e^{-1.25}$. Expected stop-loss claim conditional on claim is θ, so expected stop-loss claim=$e^{-1.25}\theta$. Premium is $2e^{-1.25}\theta$.

in fact value 0.9θ was used instead of theta, so premium is $1.8e^{-1.25}\theta$, and stop loss is really set at $1.25 \times 0.9\theta = 1.125\theta$, so expected payment on stop-loss is $e^{-1.125}\theta$. Percentage loading is therefore $\frac{1.8e^{-1.25}\theta}{e^{-1.125}\theta} - 1 = 1.8e^{-0.125} - 1 = 1.588494 - 1 = 58.85\%$.

9.4 Analytic Results

7

The severity is exponential with mean θ. The frequency is negative binomial with parameters $r = 2$ and β. The aggregate severity of n losses therefore follows a gamma distribution with $\alpha = n$. We therefore have that the probability that the aggregate loss is zero is $p_0 = \frac{1}{(1+\beta)^2}$, while if it is non-zero, the pdf of the aggregate loss is

$$\begin{aligned} f(x) &= (1+\beta)^{-2}\sum_{n=1}^{\infty}(n+1)\left(\frac{\beta}{1+\beta}\right)^n \frac{x^{n-1}}{\theta^n(n-1)!}e^{-\frac{x}{\theta}} \\ &= (1+\beta)^{-2}e^{-\frac{x}{\theta}}\frac{\beta}{\theta(1+\beta)}\sum_{n=1}^{\infty}\frac{(n+1)}{(n-1)!}\left(\frac{x\beta}{\theta(1+\beta)}\right)^{n-1} \end{aligned}$$

We have

$$\begin{aligned} \sum_{n=1}^{\infty}\frac{(n+1)a^{n-1}}{(n-1)!} &= \frac{1}{a}\frac{d}{da}\left(\sum_{n=1}^{\infty}\frac{a^{n+1}}{(n-1)!}\right) \\ &= \frac{1}{a}\frac{d}{da}\left(a^2\sum_{n=1}^{\infty}\frac{a^{n-1}}{(n-1)!}\right) \\ &= \frac{1}{a}\frac{d}{da}\left(a^2e^a\right) \\ &= (a+2)\,e^a \end{aligned}$$

Substituting this into the equation above gives

$$\begin{aligned} f(x) &= (1+\beta)^{-2}e^{-\frac{x}{\theta}}\frac{\beta}{\theta(1+\beta)}\left(\left(\frac{x\beta}{\theta(1+\beta)}\right)+2\right)e^{\left(\frac{x\beta}{\theta(1+\beta)}\right)} \\ &= (1+\beta)^{-4}\theta^{-2}\beta\,(x\beta+2)\,e^{-\frac{x}{\theta(1+\beta)}} \end{aligned}$$

This is a mixture of gamma distributions.

Alternatively, we can obtain this result using moment generating functions. For an aggregate loss model whose frequency distribution has probability generating function $P_N(z)$, and whose severity distribution has moment generating function $M_X(t)$, the aggregate loss model has moment generating function $M_S(t) = P_N(M_X(t))$.

In the case of the negative binomial gamma distibution, $P_N(z) = (1+\beta-\beta z)^{-r}$ and $M_X(t) = \left(\frac{1}{1-\theta t}\right)^{\alpha}$. This gives

$$M_S(t) = \left(1+\beta-\beta\left(\frac{1}{1-\theta t}\right)^{\alpha}\right)^{-r}$$

In the particular case $r = 2, \alpha = 1$ gives

$$\begin{aligned} M_S(t) &= \left(1+\beta-\beta\frac{1}{1-\theta t}\right)^{-2} = \left(\frac{(1+\beta)(1-\theta t)-\beta}{1-\theta t}\right)^{-2} = \left(\frac{1-\theta t}{1-(1+\beta)\theta t}\right)^{2} \\ &= \left(\frac{1}{1+\beta} + \left(\frac{\beta}{1+\beta}\right)\frac{1}{1-(1+\beta)\theta t}\right)^{2} \\ &= \frac{1}{(1+\beta)^2} + 2\frac{\beta}{(1+\beta)^2}\frac{1}{1-(1+\beta)\theta t} + \frac{\beta^2}{(1+\beta)^2}\frac{1}{(1-(1+\beta)\theta t)^2} \end{aligned}$$

This is a mixture of a point mass at 0 with probability $(1+\beta)^{-2}$; an exponential distribution with mean $(1+\beta)\theta$ with probability $\frac{2\beta}{(1+\beta)^2}$; and a gamma distribution with $\theta = (1+\beta)\theta, \alpha = 2$ with probability $\left(\frac{\beta}{(1+\beta)}\right)^2$.

8

By the general result, the compound negative binomial-exponential with $r = 15$, $\beta = 2.4$ and $\theta = 3000$ is the same a compound binomial-exponential with $n = 15$, $p = \frac{2.4}{3.4} = \frac{12}{17}$ and $\theta = 3000 \times 2.4 = 7200$.

If there are n claims, the aggregate loss follows a gamma distribution with $\alpha = n$ and $\theta = 10200$. The expected payment on the stop-loss insurance is then $10200 \int_{\frac{204000}{10200}}^{\infty} \left(x - \frac{204000}{10200}\right) \frac{x^{n-1}e^{-x}}{(n-1)!} dx$.

We have

$$\int_a^\infty \frac{x^n e^{-x}}{n!} dx = e^{-a}\left(1 + a + \frac{a^2}{2} + \ldots + \frac{a^n}{n!}\right)$$

so the expected payment on the stop-loss insurance if there are n claims is

$$\begin{aligned}
10200 \int_{20}^{\infty} (x-20) \frac{x^{n-1}e^{-x}}{(n-1)!} dx &= 7200 \left(\int_{20}^{\infty} \frac{x^n e^{-x}}{(n-1)!} dx - 20 \int_{20}^{\infty} \frac{x^{n-1}e^{-x}}{(n-1)!} dx\right) \\
&= 10200 \left(n \int_{20}^{\infty} \frac{x^n e^{-x}}{n!} dx - 20 \int_{20}^{\infty} \frac{x^{n-1}e^{-x}}{(n-1)!} dx\right) \\
&= 10200e^{-20} \left(n\left(1 + 20 + \frac{20^2}{2!} + \cdots + \frac{20^n}{n!}\right) - 20\left(1 + 20 + \frac{20^2}{2!} + \cdots + \frac{20^{n-1}}{(n-1)!}\right)\right) \\
&= 10200e^{-20} \left(n + 20(n-1) + \frac{20^2(n-2)}{2!} + \cdots + \frac{20^{n-1}}{(n-1)!}\right)
\end{aligned}$$

The overall expected payment on the stop-loss insurance is therefore

$$\begin{aligned}
&10200e^{-20} \sum_{n=1}^{15} \left(\binom{15}{n} \frac{12^n 5^{15-n}}{17^{15}}\right) \left(n + 20(n-1) + \frac{20^2(n-2)}{2!} + \cdots + \frac{20^{n-1}}{(n-1)!}\right) \\
&= 10200e^{-20} \sum_{n=1}^{15} \left(\binom{15}{n} \frac{12^n 5^{15-n}}{17^{15}}\right) \sum_{k=0}^{n} (n-k) \frac{20^k}{k!}
\end{aligned}$$

We can evaluate this sum in R using matrix operations:

```
expseries <-20^(0:14)/factorial(0:14)  #The terms in k
nvect<-dbinom(1:15,size=15,prob=12/17)  #The terms in n
nminusk<-pmax(rep(1,15)%*%t(1:15)-(1:15)*rep(1,15),0)  #the n-k term
t(expseries)%*%nminusk%*%nvect/exp(20)*10200  #The expected payment
```

This gives the expected payment on the stop-loss insurance as $137.17.

9.5 Computing the Aggregate Claims Distribution

9

ETNB $r = -0.6$ $\beta = 7$,
$a = \frac{\beta}{1+\beta} = \frac{7}{8}$, $b = \frac{(r-1)\beta}{1+\beta} = -1.4$
$q_1 = \frac{-0.6\times 7}{8(8^{-0.6}-1)} = 0.7365057$ $q_2 = \left(\frac{7}{8} - \frac{1.4}{2}\right) q_1 = 0.1288885$ $q_3 = \left(\frac{7}{8} - \frac{1.4}{3}\right) q_2 = 0.05262947$
$(n+1)\left(\frac{3}{4}\right)^n$ 0.0625 0.0625 0.09375 0.1054687890625 0.1054687890625

$p_0 = 0.0625$

$p_1 = 0.09375 * 0.7365057 = 0.06904741$

$p_2 = 0.09375 * 0.1288885 + 0.1054687890625 * 0.7365057^2 = 0.06929383$

$p_3 = 0.09375 * 0.05262947 + 0.1054687890625 * (0.7365057 * 0.1288885 * 2) + 0.1054687890625 * 0.7365057^3 = 0.06709359$

So the total probability that the aggregate loss is at most 3 is $0.0625 + 0.06904741 + 0.06929383 + 0.06709359 = 0.2679348$.

9.6 The Recursive Method

10

The recurrence formula gives $f_S(n) = \frac{\sum_{i=1}^{n}\left(a+\frac{bi}{n}\right)f_X(i)f_S(n-i)}{1-af_X(0)}$ $f_S(n) = \sum_{i=1}^{n} \frac{2.4i}{n}\binom{i+9}{i}\left(\frac{2.3}{3.3}\right)^i\left(\frac{1}{3.3}\right)^{10} f_S(n-i)$
$f_X(0) = \frac{1}{3.3^{10}}$

$$f_S(0) = e^{-2.4}\sum_{n=0}^{\infty}\frac{2.4^n}{3.3^{10n}n!} = e^{\frac{2.4}{3.3^{10}}-2.4} = 0.09071937$$

$$f_S(1) = \frac{2.4}{3.3^{10}}\left(10 \times \frac{2.3}{3.3} \times f_S(0)\right) = 0.000009907995$$

$$f_S(2) = \frac{2.4}{3.3^{10}}\left(\frac{10}{2} \times \frac{2.3}{3.3} \times f_S(1) + \frac{10 \times 11}{2} \times \left(\frac{2.3}{3.3}\right)^2 \times f_S(0)\right) = 0.00003798119$$

$$f_S(3) = \frac{2.4}{3.3^{10}}\left(\frac{10}{3} \times \frac{2.3}{3.3} \times f_S(2) + \frac{2 \times 10 \times 11}{3 \times 2} \times \left(\frac{2.3}{3.3}\right)^2 \times f_S(1) + \frac{10 \times 11 \times 12}{6} \times \left(\frac{2.3}{3.3}\right)^3 \times f_S(0)\right) = 0.0001058901$$

So the probability that the aggregate loss is at most 3 is therefore $0.09071937+0.000009907995+0.00003798119+0.0001058901 = 0.09087315$

11

For the zero-truncated ETNB distribution, we have that
$a = \frac{\beta}{1+\beta} = \frac{3}{4}$, $b = \frac{(r-1)\beta}{1+\beta} = -1.2$

$$q_1 = \frac{-0.6 \times 3}{4(4^{-0.6} - 1)} = 0.7968484$$
$$q_2 = \left(\frac{3}{4} - \frac{1.2}{2}\right) 0.7968484 = 0.119527260$$
$$q_3 = \left(\frac{3}{4} - \frac{1.2}{3}\right) 0.119527260 = 0.04183454100$$
$$q_4 = \left(\frac{3}{4} - \frac{1.2}{4}\right) 0.04183454100 = 0.018825543450$$
$$q_5 = \left(\frac{3}{4} - \frac{1.2}{5}\right) 0.018825543450 = 0.009601027159$$
$$q_6 = \left(\frac{3}{4} - \frac{1.2}{6}\right) 0.009601027159 = 0.005280564937$$
$$q_7 = \left(\frac{3}{4} - \frac{1.2}{7}\right) 0.005280564937 = 0.003055183999$$
$$q_8 = \left(\frac{3}{4} - \frac{1.2}{8}\right) 0.003055183999 = 0.001833110399$$
$$q_9 = \left(\frac{3}{4} - \frac{1.2}{9}\right) 0.001833110399 = 0.0011304180793$$
$$q_{10} = \left(\frac{3}{4} - \frac{1.2}{10}\right) 0.0011304180793 = 0.0007121633899$$
$$q_{11} = \left(\frac{3}{4} - \frac{1.2}{11}\right) 0.0007121633899 = 0.0004564319907995$$
$$q_{12} = \left(\frac{3}{4} - \frac{1.2}{12}\right) 0.0004564319907995 = 0.0002966807940196$$
$$q_{13} = \left(\frac{3}{4} - \frac{1.2}{13}\right) 0.0002966807940196 = 0.0001951246760669$$

With the deductible set at 10, the probability that a loss does not lead to a claim is $0.7968484+0.119527260+0.04183454100 + 0.018825543450 + 0.009601027159 + 0.005280564937 + 0.003055183999 + 0.001833110399 + 0.0011304180793 = 0.9981311736993669$ The distribution of the claim value is therefore

$$q_0 = 0.9981311736993669$$
$$q_1 = 0.0004564319907995$$
$$q_2 = 0.0002966807940196$$
$$q_3 = 0.0001951246760669$$

For the primary distribution $a = \frac{\beta}{1+\beta} = \frac{3}{8}$ and $b = \frac{(r-1)\beta}{1+\beta} = -0.3$.
Now we can use the recursive formula

$$f_S(n) = \frac{\sum_{i=1}^{n} \left(0.375 - \frac{0.3i}{n}\right) q_i f_S(n-i)}{1 - \frac{3}{8} \times 0.99813}$$

We calculate

$$\begin{aligned} f_S(0) &= \sum_{n=0}^{\infty} p_n (f_X(0))^n = \sum_{n=0}^{\infty} \binom{n-0.8}{n} \left(\frac{3}{8}\right)^n \left(\frac{5}{8}\right)^{0.2} (f_X(0))^n \\ &= \left(\frac{5}{8}\right)^{0.2} \left(1 - \frac{3}{8} f_X(0)\right)^{-0.2} = 0.9997759 \end{aligned}$$

Now using the recurrence, we get

$$f_S(1) = \frac{0.075 \times 0.000456 \times 0.9998}{0.6257008} = 0.00005469823$$

$$f_S(2) = \frac{0.225 \times 0.000456 \times 0.0000547 + 0.075 \times 0.000297 \times 0.9998}{0.6257008} = 0.00003556283$$

$$f_S(3) = \frac{0.275 \times 0.000456 \times 0.0000356 + 0.175 \times 0.000297 \times 0.0000547 + 0.075 \times 0.000195 \times 0.9998}{0.6257008} = 0.00002339517$$

The probability of paying out at least \$400 to a single driver is therefore $1 - 0.9997759 - 0.00005469823 - 0.00003556283 - 0.00002339517 = 0.0001104438$

9.6.2 Applications to Compound Frequency Models

12

The zero-truncated logarithmic distribution has $a = 0.8$, $b = -0.8$. This gives $p_1\left(\sum_{n=0}^{\infty}\frac{0.8^n}{n+1}\right) = 1$ $p_1 = -\frac{0.8}{\log(1-0.8)} = 0.4970679$.

$$p_1 = 0.4970679$$
$$p_2 = 0.1988272$$
$$p_3 = 0.1060412$$

Now we compound with a Poisson with $\lambda = 0.1$. The probability of the total being 0 is $e^{-0.1}$ (since the secondary distribution is zero-truncated). The recurrence is

$$f_S(n) = \sum_{i=1}^{n}\frac{0.1i}{n}p_i f_S(n-i)$$

So we calculate:

$$f_S(1) = 0.1 \times 0.497 \times e^{-0.1} = 0.04970679e^{-0.1}$$
$$f_S(2) = 0.05 \times 0.497 \times 0.04970679e^{-0.1} + 0.1 \times 0.199e^{-0.1} = 0.0211181e^{-0.1}$$
$$f_S(3) = (0.0333 \times 0.497 \times 0.0211 + 0.0667 \times 0.199 \times 0.0497 + 0.1 \times 0.106)\,e^{-0.1} = 0.010705e^{-0.1}$$

Now for the overall compound distribution, we have $f_A(0) = e^{-6}\sum_{n=0}^{\infty}\frac{6^n}{n!}e^{-0.1n} = e^{6e^{-0.1}-6} = e^{-0.5709755} = 0.564974$.

The recurrence is

$$f_A(n) = \sum_{i=1}^{n}\frac{6i}{n}f_S(i)f_A(n-i)$$

So we calculate:

$$f_A(1) = 6 \times 0.0497e^{-0.1} \times 0.564974 = 0.1524635$$
$$f_A(2) = 3 \times 0.04970679e^{-0.1} \times 0.1524635 + 6 \times 0.0211181e^{-0.1} \times 0.564974 = 0.08534651$$
$$f_A(3) = 2 \times 0.04970679e^{-0.1} \times 0.08534651 + 4 \times 0.0211181e^{-0.1} \times 0.1524635 + 6 \times 0.010705e^{-0.1} \times 0.564974 = 0.05216554$$

So the probability that the total claimed is more than 3000 is

$$1 - 0.564974 - 0.1524635 - 0.08534651 - 0.05216554 = 0.1450505$$

9.6.2 Underflow Problems

13

The recurrence is

$$f_S(n) = \sum_{i=1}^{n} \frac{\lambda i}{n} \binom{n+3}{n} 0.6875^i 0.3125^4 f_S(n-i)$$

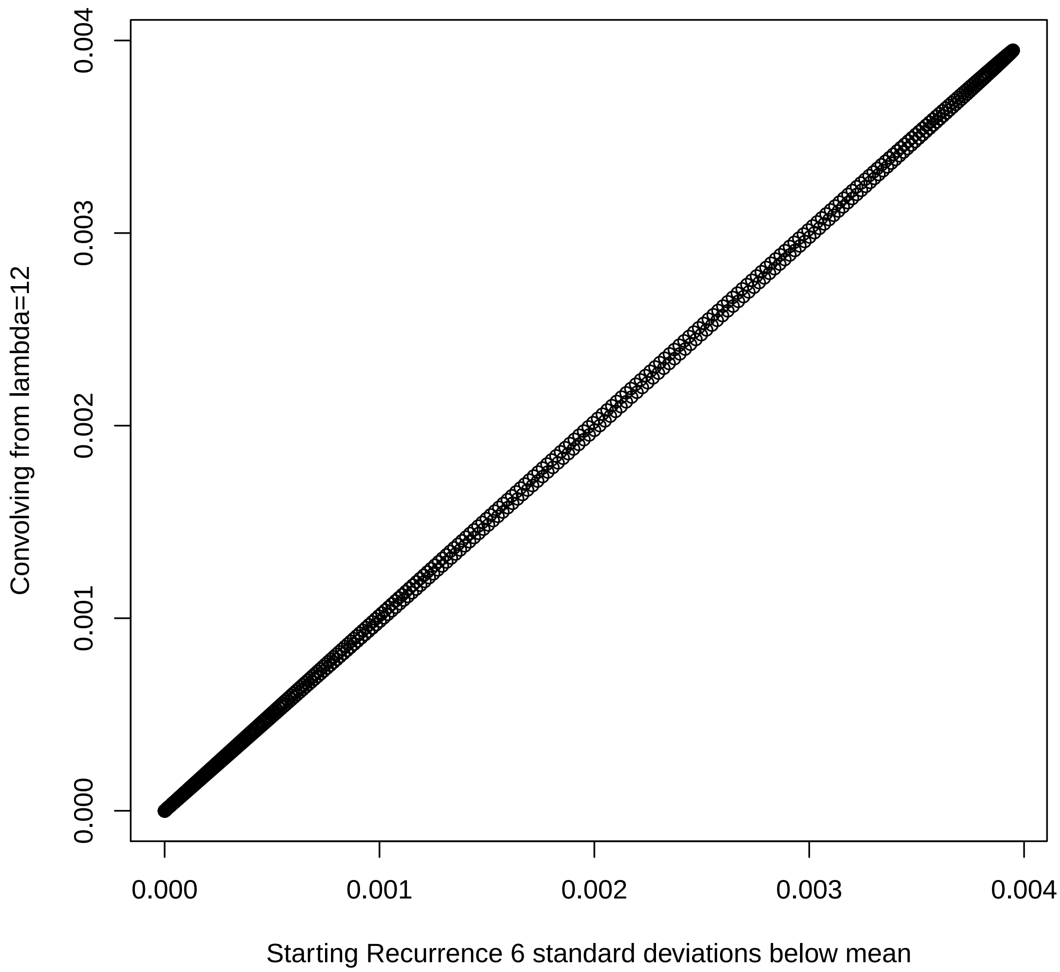
(a)

The mean of the distribution is $96 \times 8.8 = 844.8$, and the variance is $96 \times 28.16 + 96 \times 8.8^2 = 48 \times (28.16 + 77.44) = 96 \times 105.6 = 10137.6$ The standard deviation is therefore $\sqrt{10137.6} = 100.6856$, so six standard deviations below the mean is $422.4 - 6 \times 100.6856 = 240.6864$. We will start the recurrence at 241. If we assume that $f_s(240) = 0$ and $f_2(241) = 1$, then we can calculate the values

$$f_S(n) = \sum_{i=1}^{n} \frac{96i}{n} \frac{(i+1)(i+2)(i+3)}{6} 0.6875^i 0.3125^4 f_S(n-i)$$

(b) Solution for $\lambda = 12$:

$$f_S(n) = \sum_{i=1}^{n} \frac{12i}{n} \frac{(i+1)(i+2)(i+3)}{6} 0.6875^i 0.3125^4 f_S(n-i)$$


0.000
0.001
0.002
0.003
0.004
Convolving from lambda=12
0.000
0.001
0.002
0.003
0.004
Starting Recurrence 6 standard deviations below mean

9.6.3 Numerical Stability

14

For the binomial distribution $a = -\frac{p}{1-p}$ $b = (n+1)\frac{p}{1-p}$
The recurrence relation is

$$\begin{aligned} f_S(x) &= \frac{1}{1 - af_X(0)} \left(\sum_{y=1}^{x} \left(a + b\frac{y}{x} \right) f_x(y) f_s(x-y) \right) \\ &= \frac{1}{1 + \frac{p}{1-p} f_X(0)} \frac{p}{1-p} \left(\sum_{y=1}^{x} \left((n+1)\frac{y}{x} - 1 \right) f_x(y) f_s(x-y) \right) \\ &= \frac{p}{1 - p + pf_X(0)} \left(\sum_{y=1}^{x} \left((n+1)\frac{y}{x} - 1 \right) f_x(y) f_s(x-y) \right) \end{aligned}$$

Substituting $p = 0.8$, $n = 7$ and $f_x(0) = 0.21$, $f_x(1) = 0.41$ and $f_x(3) = 0.38$, this becomes

$$\begin{aligned} f_S(x) &= \frac{0.8}{1 - 0.8 + 0.8 \times 0.21} \left(0.41 \left(\frac{n+1}{x} - 1 \right) f_s(x-1) + 0.38 \left(3\frac{n+1}{x} - 1 \right) f_s(x-3) \right) \\ &= \frac{50}{23} \left(0.41 \left(\frac{n+1}{x} - 1 \right) f_s(x-1) + 0.38 \left(3\frac{n+1}{x} - 1 \right) f_s(x-3) \right) \\ &= \left(\frac{41}{46} \left(\frac{n+1}{x} - 1 \right) f_s(x-1) + \frac{38}{46} \left(3\frac{n+1}{x} - 1 \right) f_s(x-3) \right) \end{aligned}$$

9.6.4 Continuous Severity

9.6.5 Constructing Arithmetic Distributions

15

(a) Using the method of rounding, we have
$p_0 = 1 - e^{-\frac{1}{2\theta}}$ $p_n = e^{-\frac{2n-1}{2\theta}} \left((1 - e^{-\frac{1}{\theta}}\right)$
This is a zero modified geometric distribution.
(b) On the interval $[a, a+2]$, we have

$$\begin{aligned}
\frac{1}{\theta}\int_a^{a+2} e^{-\frac{x}{\theta}}\,d\theta &= e^{-a}(1-e^{-\frac{2}{\theta}})\\
\frac{1}{\theta}\int_a^{a+2} xe^{-\frac{x}{\theta}}\,d\theta &= e^{-a}\left(a(1-e^{-\frac{2}{\theta}}) + \int_0^2 \frac{t}{\theta}e^{-\frac{t}{\theta}}\,dt\right)\\
&= e^{-a}\left(a(1-e^{-\frac{2}{\theta}}) + \left[-te^{-\frac{t}{\theta}}\right]_0^2 + \int_0^2 e^{-\frac{t}{\theta}}\,dt\right)\\
&= e^{-a}\left(a(1-e^{-\frac{2}{\theta}}) - 2e^{-\frac{2}{\theta}} + \theta\left(1-e^{-\frac{2}{\theta}}\right)\right)\\
&= e^{-a}\left(a+\theta-e^{-\frac{2}{\theta}}(a+\theta+2)\right)\\
\frac{1}{\theta}\int_a^{a+2} x^2e^{-\frac{x}{\theta}}\,d\theta &= e^{-a}\int_0^2 (a^2+2at+t^2)\frac{e^{-\frac{t}{\theta}}}{\theta}\,dt\\
&= e^{-a}\left(a^2(1-e^{-\frac{2}{\theta}}) + 2a\left(\theta - e^{-\frac{2}{\theta}}(\theta+2)\right) + \int_0^2 t^2\frac{e^{-\frac{t}{\theta}}}{\theta}\,dt\right)\\
&= e^{-a}\left((a^2+2a\theta)(1-e^{-\frac{2}{\theta}}) - 4ae^{-\frac{2}{\theta}} + \left[-t^2e^{-\frac{t}{\theta}}\right]_0^2 + \left[-2t\theta e^{-\frac{t}{\theta}}\right]_0^2 + \int_0^2 2\theta e^{-\frac{t}{\theta}}\,dt\right)\\
&= e^{-a}\left((a^2+2a\theta)(1-e^{-\frac{2}{\theta}}) - 4ae^{-\frac{2}{\theta}} - 4e^{-\frac{2}{\theta}} - 4\theta e^{-\frac{2}{\theta}} + 2\theta^2(1-e^{-\frac{2}{\theta}})\right)
\end{aligned}$$

Therefore, matching moments on this interval gives us

$$\begin{aligned}
p_a + p_{a+1} + p_{a+2} &= e^{-a}(1-e^{-\frac{2}{\theta}})\\
ap_a + (a+1)p_{a+1} + (a+2)p_{a+2} &= e^{-a}\left(a+\theta-e^{-\frac{2}{\theta}}(a+\theta+2)\right)\\
a^2p_a + (a+1)^2p_{a+1} + (a+2)^2p_{a+2} &= e^{-a}\left((a^2+2a\theta+2\theta^2)(1-e^{-\frac{2}{\theta}}) - 4e^{-\frac{2}{\theta}}(a+\theta+1)\right)
\end{aligned}$$

We solve these:

$$\begin{aligned}
p_{a+1} + 2p_{a+2} &= e^{-a}\left(\theta - e^{-\frac{2}{\theta}}(\theta+2)\right) \\
(a+1)p_{a+1} + 2(a+2)p_{a+2} &= e^{-a}\left((a\theta + 2\theta^2)(1 - e^{-\frac{2}{\theta}}) + 2ae^{-\frac{2}{\theta}} - 4e^{-\frac{2}{\theta}}(a+\theta+1)\right) \\
2p_{a+2} &= e^{-a}\left(2(a+1)e^{-\frac{2}{\theta}} - 2ae^{-\frac{2}{\theta}} - 4e^{-\frac{2}{\theta}} - 4\theta e^{-\frac{2}{\theta}} + \theta^2(1 - e^{-\frac{2}{\theta}})\right) \\
&= e^{-a}\left(\theta^2 - (\theta+2)^2 e^{-\frac{2}{\theta}}\right) \\
p_{a+1} &= e^{-a}\left(\theta - e^{-\frac{2}{\theta}}(\theta+2) - \left(\theta^2 - (\theta+2)^2 e^{-\frac{2}{\theta}}\right)\right) \\
&= e^{-a}\left(\theta(1-\theta) + e^{-\frac{2}{\theta}}(\theta+2)(\theta+1)\right) \\
p_a &= e^{-a}\left(1 - e^{-\frac{2}{\theta}} - \theta(1-\theta) - e^{-\frac{2}{\theta}}(\theta+2)(\theta+1) - \frac{1}{2}\left(\theta^2 - (\theta+2)^2 e^{-\frac{2}{\theta}}\right)\right) \\
&= \frac{e^{-a}}{2}\left(2 - 2\theta + \theta^2 - e^{-\frac{2}{\theta}}(2 - \theta(\theta+2))\right)
\end{aligned}$$

Thus for an odd number

$$p_{2n+1} = e^{-2n}\left(\theta(1-\theta) + e^{-\frac{2}{\theta}}(\theta+2)(\theta+1)\right)$$

and for an even number

$$\begin{aligned}
p_{2n} &= \frac{e^{-2n}}{2}\left(2 - 2\theta + \theta^2 - e^{-\frac{2}{\theta}}(2 - \theta(\theta+2))\right) + \frac{e^{-2(n-1)}}{2}\left(\theta^2 - (\theta+2)^2 e^{-\frac{2}{\theta}}\right) \\
&= \frac{e^{-2(n-1)}}{2}\left(\theta^2 - e^{-\frac{2}{\theta}}\left((\theta+2)^2 - (2 - 2\theta + \theta^2)\right) - e^{-\frac{4}{\theta}}(2 - \theta(\theta+2))\right) \\
&= \frac{e^{-2(n-1)}}{2}\left(\theta^2 - e^{-\frac{2}{\theta}}(6\theta+2) + e^{-\frac{4}{\theta}}(\theta^2 + 2\theta - 2)\right)
\end{aligned}$$

9.7 Individual Policy Modifications

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Conditional on n losses, aggregate loss follows gamma distribution with $\theta = 2000$ and $r = n$.

(a)

We previously showed that the compound negative binomial-exponential distribution is the same as a compound binomial exponential distribution with $n = r$, $p = \frac{\beta}{1+\beta}$ and $\theta = \theta\beta$, which in this case gives $n = 4$, $p = \frac{2.1}{3.1}$ and $\theta = 2.1 \times 3000 = 6200$. This gives

$$f(x) = \frac{2.1}{12000 \times 3.1^5}\left(\left(\frac{2.1x}{6200}\right)^3 + 12\left(\frac{2.1x}{6200}\right)^2 + 36\left(\frac{2.1x}{6200}\right) + 18\right)e^{-\frac{x}{6200}}$$

The probability mass at zero is $\frac{1}{3.1^4}$.

(b) With a deductible of \$500, the distribution per loss has probability mass $1 - e^{-0.25}$ at zero. The distribution of the number of claims is negative binomial with $r = 4$ and $\beta = 2.1e^{-0.25}$. If there is a claim, by the memoryless property of the exponential distribution, severity has the same distribution as in part (a), so the aggregate loss is compound negative-binomial exponential with $r = 4$, $\beta = 2.1e^{-0.25}$ and $\theta = 3000$. This is the same distribution as compound binomial exponential with $n = 4$, $p = \frac{2.1e^{-0.25}}{1+2.1e^{-0.25}}$ and $\theta = 6200e^{-0.25}$, so the probability mass at zero is $\frac{1}{(1+2.1e^{-0.25})^4}$. The density is

$$f(x) = \left(\frac{2.1e^{-0.25}}{12000 \times (1 + 2.1e^{-0.25})^5}\right)e^{-\frac{x}{2000(1+2.1e^{-0.25})}}$$
$$\left(\left(\frac{2.1e^{-0.25}x}{2000(1 + 2.1e^{-0.25})}\right)^3 + 12\left(\frac{2.1e^{-0.25}x}{2000(1 + 2.1e^{-0.25})}\right)^2 + 36\left(\frac{2.1e^{-0.25}x}{2000(1 + 2.1e^{-0.25})}\right) + 18\right)$$

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We will use a normal approximation for the aggregate loss. The mean of the aggregate loss distribution is $100 \times 6800 = 680000$. The variance is $100 \times 6800^2 + 100 \times 13600000 = 5984000000$. If we use a normal approximation, then it has mean $\mu = 680000$ and standard deviation $\sigma = 77356.32$. The expected payment on the stop loss insurance is then

$$\begin{aligned}
\int_{1000000}^{\infty} \frac{(x-1000000)e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma}\,dx &= \frac{1}{\sqrt{2\pi}\sigma}\int_{1000000}^{\infty}(x-\mu)e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx - (1000000-\mu)\int_{1000000}^{\infty}\frac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma}\,dx\\
&= \frac{1}{\sqrt{2\pi}\sigma}\left[-\sigma^2 e^{-\frac{(x-\mu)^2}{2\sigma^2}}\right]_{1000000}^{\infty} - (1000000-\mu)\left(1-\Phi\left(\frac{1000000-\mu}{\sigma}\right)\right)\\
&= \frac{e^{-\frac{(1000000-\mu)^2}{2\sigma^2}}\sigma}{\sqrt{2\pi}} - (1000000-\mu)\left(1-\Phi\left(\frac{1000000-\mu}{\sigma}\right)\right)\\
&= 0.2989764
\end{aligned}$$

The expected square of the payment is

$$\begin{aligned}
&\int_{1000000}^{\infty} \frac{(x-1000000)^2 e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma}\,dx\\
&= \frac{1}{\sqrt{2\pi}\sigma}\int_{1000000}^{\infty}(x-\mu)^2 e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx - 2(1000000-\mu)\int_{1000000}^{\infty}(x-\mu)\frac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma}\,dx + (1000000-\mu)^2\int_{1000000}^{\infty}\frac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma}\,dx\\
&= \frac{1}{\sqrt{2\pi}\sigma}\left[-\sigma^2(x-\mu)e^{-\frac{(x-\mu)^2}{2\sigma^2}}\right]_{1000000}^{\infty} + \int_{1000000}^{\infty}\frac{\sigma e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}}\,dx - 2(1000000-\mu)\frac{1}{\sqrt{2\pi}\sigma}\left[-\sigma^2 e^{-\frac{(x-\mu)^2}{2\sigma^2}}\right]_{1000000}^{\infty}\\
&\quad + (1000000-\mu)^2\left(1-\Phi\left(\frac{1000000-\mu}{\sigma}\right)\right)\\
&= \left((1000000-\mu)^2+\sigma^2\right)\left(1-\Phi\left(\frac{1000000-\mu}{\sigma}\right)\right) - \frac{(1000000-\mu)\sigma e^{-\frac{(1000000-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}}\\
&= 9745.92
\end{aligned}$$

The standard deviation is therefore

$$\sqrt{9745.92 - 0.2989764^2} = \sqrt{9745.83061311} = 98.7209735219$$

The premium is therefore $98.7209735219 + 0.2989764 = \$99.02$.

(b) With a deductible of \$1,000, the expected claim per loss is

$$\begin{aligned}
\int_{1000}^{\infty}\frac{(x-1000)x^{2.4}e^{-\frac{x}{2000}}}{2000^{3.4}\Gamma(3.4)}\,dx &= \int_{1000}^{\infty}\frac{x^{3.4}e^{-\frac{x}{2000}}}{2000^{3.4}\Gamma(3.4)}\,dx - 1000\int_{1000}^{\infty}\frac{x^{2.4}e^{-\frac{x}{2000}}}{2000^{3.4}\Gamma(3.4)}\,dx\\
&= 6800\int_{1000}^{\infty}\frac{x^{3.4}e^{-\frac{x}{2000}}}{2000^{4.4}\Gamma(4.4)}\,dx - 1000\int_{1000}^{\infty}\frac{x^{2.4}e^{-\frac{x}{2000}}}{2000^{3.4}\Gamma(3.4)}\,dx\\
&= 5801.558
\end{aligned}$$

the expected squared payment per loss is

$$\int_{1000}^{\infty}\frac{(x-1000)^2x^{2.4}e^{-\frac{x}{2000}}}{2000^{3.4}\Gamma(3.4)}\,dx=\int_{1000}^{\infty}\frac{x^{4.4}e^{-\frac{x}{2000}}}{2000^{3.4}\Gamma(3.4)}\,dx-2000\int_{1000}^{\infty}\frac{x^{3.4}e^{-\frac{x}{2000}}}{2000^{3.4}\Gamma(3.4)}\,dx+1000000\int_{1000}^{\infty}\frac{x^{2.4}e^{-\frac{x}{2000}}}{2000^{3.4}\Gamma(3.4)}\,dx$$

$$=59840000\int_{1000}^{\infty}\frac{x^{4.4}e^{-\frac{x}{2000}}}{2000^{5.4}\Gamma(5.4)}\,dx-13600000\int_{1000}^{\infty}\frac{x^{3.4}e^{-\frac{x}{2000}}}{2000^{4.4}\Gamma(4.4)}\,dx+1000000\int_{1000}^{\infty}\frac{x^{2.4}e^{-\frac{x}{2000}}}{2000^{3.4}\Gamma(3.4)}\,d$$

$$=47239394$$

The mean of the aggregate loss is therefore $100\times5801.558=580155.8$ and the variance is $100\times5801.558^2+100\times(47239394-5801.558^2)=4723939400$. The standard deviation is therefore $\sqrt{4723939400}=68730.92$.

$$\int_{1000000}^{\infty}\quad\frac{e^{-\frac{(1000000-\mu)^2}{2\sigma^2}}\sigma}{\sqrt{2\pi}}-(1000000-\mu)\left(1-\Phi\left(\frac{1000000-\mu}{\sigma}\right)\right)$$

$$=0.000005388222$$

The expected square of the payment is

$$\left((1000000-\mu)^2+\sigma^2\right)\left(1-\Phi\left(\frac{1000000-\mu}{\sigma}\right)\right)-\frac{(1000000-\mu)\sigma e^{-\frac{(1000000-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}}$$

$$=0.1129654$$

The variance is therefore

$$\left((1000000-\mu)^2+\sigma^2\right)\left(1-\Phi\left(\frac{1000000-\mu}{\sigma}\right)\right)-\frac{(1000000-\mu)\sigma e^{-\frac{(1000000-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}}-\left(\frac{e^{-\frac{(1000000-\mu)^2}{2\sigma^2}}\sigma}{\sqrt{2\pi}}-(1000000-\mu)\left(1-\Phi\left(\right.\right.\right.$$

$$=0.1129654$$

The premium is therefore \$0.3361086

9.8 Individual Risk Model

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We will use a normal approximation for the first three types of workers, then treat the senior managers separately. We have the following for the first 3 types of workers:

Type of Worker	$\mathbb{E}(N)$	$\mathrm{Var}(N)$	$\mathbb{E}(S)$ (millions)	$\mathrm{Var}(S)$ $\times 10^{10}$
Manual Labourer	46.22	45.7578	4.622	45.7578
Administrator	7.08	7.06584	0.6372	5.7233304
Manager	8.02	7.9398	1.604	31.7592
Total			6.8632	83.2403304

Thus the aggregate losses for the first three groups can be approximated by a normal distribution with mean \$6,863,200 and standard deviation $\sqrt{832403304000} = \$912,361.388924$. We find the probability that the aggregate losses exceed \$10,000,000 by conditioning on the number of senior managers who die.

We can consider the various cases in a table

Senior Managers	Probability	Z-statistic	Probability aggregate more than 10,000,000	P
0	4.832131×10^{-01}	3.4381113	0.0002928934	1.415300×10^{-04}
1	3.550137×10^{-01}	2.3420544	0.0095889599	3.404212×10^{-03}
2	1.267906×10^{-01}	1.2459975	0.1063826587	1.348832×10^{-02}
3	2.932572×10^{-02}	0.1499406	0.4404057460	1.291522×10^{-02}
4	4.937494×10^{-03}	-0.9461163	0.8279553694	4.088025×10^{-03}
5	6.448972×10^{-04}	-2.0421732	0.9794328242	6.316334×10^{-04}
6	6.799936×10^{-05}	-3.1382301	0.9991501431	6.794157×10^{-05}
7	5.947466×10^{-06}	-4.2342870	0.9999885361	5.947398×10^{-06}
8	4.399911×10^{-07}	-5.3303439	0.9999999510	4.399911×10^{-07}
9	2.793594×10^{-08}	-6.4264008	0.9999999999	2.793594×10^{-08}
10	1.539327×10^{-09}	-7.5224577	1.0000000000	1.539327×10^{-09}
11	7.425327×10^{-11}	-8.6185146	1.0000000000	7.425327×10^{-11}
12	3.157027×10^{-12}	-9.7145715	1.0000000000	3.157027×10^{-12}
13	1.189461×10^{-13}	10.8106285	1.0000000000	1.189461×10^{-13}
14	3.987988×10^{-15}	11.9066854	1.0000000000	3.987988×10^{-15}
15	1.193683×10^{-16}	13.0027423	1.0000000000	1.193683×10^{-16}
16	3.197366×10^{-18}	14.0987992	1.0000000000	3.197366×10^{-18}
17	7.676750×10^{-20}	15.1948561	1.0000000000	7.676750×10^{-20}
18	1.653722×10^{-21}	16.2909130	1.0000000000	1.653722×10^{-21}
19	3.197313×10^{-23}	17.3869699	1.0000000000	3.197313×10^{-23}
20	5.546360×10^{-25}	18.4830268	1.0000000000	5.546360×10^{-25}
21	8.624078×10^{-27}	19.5790837	1.0000000000	8.624078×10^{-27}
22	1.200011×10^{-28}	20.6751406	1.0000000000	1.200011×10^{-28}
23	1.490697×10^{-30}	21.7711975	1.0000000000	1.490697×10^{-30}
24	1.647879×10^{-32}	22.8672544	1.0000000000	1.647879×10^{-32}
25	1.614249×10^{-34}	23.9633113	1.0000000000	1.614249×10^{-34}
26	1.393778×10^{-36}	25.0593682	1.0000000000	1.393778×10^{-36}
27	1.053498×10^{-38}	26.1554251	1.0000000000	1.053498×10^{-38}
28	6.910704×10^{-41}	27.2514820	1.0000000000	6.910704×10^{-41}
29	3.890614×10^{-43}	28.3475389	1.0000000000	3.890614×10^{-43}
30	1.852674×10^{-45}	29.4435958	1.0000000000	1.852674×10^{-45}
31	7.318000×10^{-48}	30.5396527	1.0000000000	7.318000×10^{-48}
32	2.333546×10^{-50}	31.6357096	1.0000000000	2.333546×10^{-50}
33	5.772531×10^{-53}	32.7317665	1.0000000000	5.772531×10^{-53}
34	1.039471×10^{-55}	33.8278234	1.0000000000	1.039471×10^{-55}
35	1.212212×10^{-58}	34.9238803	1.0000000000	1.212212×10^{-58}
36	6.871948×10^{-62}	36.0199372	1.0000000000	6.871948×10^{-62}

Total probability 0.0347433.

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The mean aggregate loss is $6863200 + 720000 = \$7,583,200$, and the variance of the aggregate loss is $832403304000 + 705600000000 = 1538003304000$, (so the standard deviation is 1240162.61192)

(a) Using a normal distribution, the probability that the aggregate loss exceeds 10,000,000 is $1-\Phi\left(\frac{10000000-7583200}{1240162.61192}\right) = 1-\Phi(1.94877669813) = 0.02566105$.

(b) Using a gamma distribution, we have $\theta = \frac{1538003304000}{7583200} = 202817.188522$ and $\alpha = \frac{7583200}{202817.188522} = 37.3893359594$ We are trying to calculate the probability that the distribution is more than 49.30548575573θ, which is given by $\frac{\int_{49.3054857573}^{\infty} x^{\alpha-1}e^{-x}\,dx}{\Gamma(\alpha)} = 0.03494672$

(c) Using a log-normal distribution, the mean of a log-normal distribution is $e^{\mu+\frac{\sigma^2}{2}}$, while the variance is $e^{2\mu+\sigma^2}(e^{\sigma^2}-1)$. We therefore have $e^{\sigma^2}-1 = \frac{1538003304000}{7583200^2} = 1.02674559401$, so $\sigma^2 = \log(2.02674559401) = 0.706431350964$. This gives $e^{\mu} = \frac{7583200}{\sqrt{2.02674559401}} = 5326634.42448$, so $\mu = 15.4882301566$

The probability that this is greater than 10,000,000 is therefore $1-\Phi\left(\frac{\log(10000000)-15.4882301566}{\sqrt{0.706431350964}}\right) = 1-\Phi(0.74939852665) = 0.2268085$.

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(a)

Type of worker	number n	λ	$n\lambda$	death benefit
Manual Laborer	4,622	0.01	46.22	\$100,000
Administrator	3,540	0.002	7.08	\$90,000
Manager	802	0.01	8.02	\$200,000
Senior Manager	36	0.02	0.72	\$1,000,000

The total expected number of claims is $46.22 + 7.08 + 8.02 + 0.72 = 62.04$. The distribution of a random claim amount is therefore

$$f_X(x) = \begin{cases} \frac{7.08}{62.04} = 0.114119922631 & x = 90000 \\ \frac{46.22}{62.04} = 0.745003223727 & x = 100000 \\ \frac{8.02}{62.04} = 0.129271437782 & x = 200000 \\ \frac{0.72}{62.04} = 0.0116054158607 & x = 1000000 \end{cases}$$

We will measure the total loss in units of \$10,000. We are estimating the probability that the aggregate loss exceeds 1000. Under the compound Poisson distribution, the recurrence relation is

$$f_S(x) = \sum_{y=1}^{x} 62.04\frac{y}{x}f_X(y)f_S(x-y) = \frac{1}{x}\left(63.72f_S(x-9) + 4622f_S(x-10) + 160.4f_S(x-20) + 72f_S(x-100)\right)$$

We will start this recurrence 6 standard deviations below the mean. That is at $6863200-6\times 912361.388924 = 1389031.66646$, or $x = 139$.

We now compute the recurrence in R:

```
fS<-rep(0,10000)
fS[139]<-1
for(x in 140:10000){
  fS[x]<-(63.72*fS[x-9]+462.2*fS[x-10]+160.4*fS[x-20]+72*fS[x-100])/x
}
sum(fS[1000:10000])/sum(fS)
```

This gives the probability is 0.03766295.

(b) Here we set

$$\lambda_i = \begin{cases} -\log(1-0.01) = 0.010050336 & \text{if } 1 \leqslant i \leqslant 4622 \\ -\log(1-0.002) = 0.002002003 & \text{if } 4623 \leqslant i \leqslant 8162 \\ -\log(1-0.01) = 0.010050336 & \text{if } 8163 \leqslant i \leqslant 8964 \\ -\log(1-0.02) = 0.020202707 & \text{if } 8965 \leqslant i \leqslant 9000 \end{cases}$$

Type of worker	number n	λ	$n\lambda$	death benefit
Manual Laborer	4,622	0.010050336	46.452652992	\$100,000
Administrator	3,540	0.002002003	7.08709062	\$90,000
Manager	802	0.010050336	8.060369472	\$200,000
Senior Manager	36	0.020202707	0.727297452	\$1,000,000

The total expected number of claims is $46.452652992+7.08709062+8.060369472+0.727297452 = 62.327410536$

The distribution of a random claim amount is therefore

$$f_X(x) = \begin{cases} \frac{46.452652992}{62.327410536} = 0.745300544215 & x = 90000 \\ \frac{7.08709062}{62.327410536} = 0.113707445232 & x = 100000 \\ \frac{8.060369472}{62.327410536} = 0.129323028226 & x = 200000 \\ \frac{0.727297452}{62.327410536} = 0.0116689823265 & x = 1000000 \end{cases}$$

We will measure the total loss in units of \$10,000. We are estimating the probability that the aggregate loss exceeds 1000. Under the compound Poisson distribution, the recurrence relation is

$$f_S(x) = \sum_{y=1}^{x} 62.327410536 \frac{y}{x} f_X(y) f_S(x-y) = \frac{1}{x}\left(63.78381558 f_S(x-9) + 464.52652992 f_S(x-10) + 161.20738944 f_S(x-20) + 72.72\right.$$

As in (a), we will start the recurrence at $x = 139$.

We now compute the recurrence:

```
fS<-rep(0,10000)
fS[139]<-1
for(x in 140:10000){
  fS[x]<-(63.78381558*fS[x-9]+464.52652992*fS[x-10]+161.20738944*fS[x-20]+72.7297452*fS[x-
}
sum(fS[1000:10000])/sum(fS)
```

This gives probability 0.04020969.

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Mean total loss $= 800 \times 0.02 \times 3000 + 2100 \times 0.05 \times 4000 + 500 \times 0.12 \times 5000 = 48000 + 420000 + 300000 =$ $\$768,000$. The variance of the total loss is

$$800 \times 0.02 \times 0.98 \times 3000^2 + 800 \times 0.02^2 + 2100 \times 0.05 \times 0.95 \times 4000^2 + 2100 \times 0.05 \times 1600^2 + 500 \times 0.12 \times 0.88 \times 5000^2 + 500 \times 0.12 \times 150$$

$$= 177,120,000 + 1,864,800,000 + 1,455,000,000 = 3,496,920,000$$

(a) The gamma approximation therefore has $\theta = \frac{3496920000}{768000} = \frac{1165640}{256} = 4553.28125$ and $\alpha = \frac{768000}{4553.28125} =$ 168.669572080573.

We get $\frac{800000}{\theta} = 175.697470917264$

The expected payment on the stop-loss insurance is therefore

$$\frac{\theta}{\Gamma(\alpha)} \int_{175.697470917264}^{\infty} (x^{\alpha} - 175.697470917264 x^{\alpha-1}) e^{-x} dx = \$11,234.2$$

The expected square of the payment on the stop-loss insurance is therefore

$$\frac{\theta^2}{\Gamma(\alpha)} \int_{175.697470917264}^{\infty} (x^{\alpha+1} - 2 \times 175.697470917264 x^{\alpha} + 175.697470917264^2 x^{\alpha-1}) e^{-x} dx = 740555835$$

so the variance of the stop-loss payment is 614348585, and the standard deviation is $\$24,786.06$

The reinsurance premium is therefore \$36,020.26.

(b) The normal approximation has $\mu = 768000$ and $\sigma^2 = 3496920000$, so the standard deviation is 59134.761350664128 and the cut-off for the stop-loss is 0.541136875656 standard deviations above the mean. The expected payment of the stop-loss is therefore

$$59134.761350664128 \frac{\int_{0.541136875656}^{\infty} (x - 0.541136875656) e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}$$

$$= 59134.761350664128 \left(\frac{[e^{-\frac{x^2}{2}}]_{0.541136875656}^{\infty}}{\sqrt{2\pi}} - 0.541136875656(1 - \Phi(0.541136875656)) \right)$$

$$= 59134.761350664128 \left(\frac{e^{-\frac{0.541136875656^2}{2}}}{\sqrt{2\pi}} - 0.541136875656(1 - \Phi(0.541136875656)) \right) = 10963.59$$

The expected square of the payment is

$$59134.761350664128^2 \frac{\int_{0.541136875656}^{\infty}(x-0.541136875656)^2 e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}$$

$$= 59134.761350664128^2 \frac{\int_{0.541136875656}^{\infty}(x^2-1.082273751312x+0.292829118194)e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}$$

$$= 59134.761350664128\left(\frac{[e^{-\frac{x^2}{2}}]_{0.541136875656}^{\infty}}{\sqrt{2\pi}} - 0.541136875656(1-\Phi(0.541136875656))\right)$$

$$= 59134.761350664128\left(\frac{e^{-\frac{0.541136875656^2}{2}}}{\sqrt{2\pi}} - 0.541136875656(1-\Phi(0.541136875656))\right) = 10963.59$$

Now we have that

$$\int_a^{\infty} x^2 e^{-\frac{x^2}{2}}\,dx = [-xe^{-\frac{x^2}{2}}]_a^{\infty} + \int_a^{\infty} e^{-\frac{x^2}{2}}\,dx = ae^{-\frac{a^2}{2}} + \sqrt{2\pi}(1-\Phi(a))$$

So the variance is

$$59134.76135^2\left(\frac{0.5411^{-\frac{0.5411^2}{2}}}{\sqrt{2\pi}} + 1 - \Phi(0.5411) - 1.0823\frac{e^{-\frac{0.5411^2}{2}}}{\sqrt{2\pi}} + 0.2928(1-\Phi(0.5411))\right) = 677982038.478915225279$$

The standard deviation is 26038.088226267980, so the premium is $10963.59 + 26038.088226267980 = \$37,001.68$.

22

if 20% are smokers, the expected number of claims per policy is $0.2 \times 0.02 + 0.8 \times 0.01 = 0.012$, so the premium is set to $1.1 \times 12 = 1.32$. If 30% are smokers, the expected number of claims is per policy is 0.013. The variance of the number of claims is $0.3 \times 0.02 \times 0.98 + 0.7 \times 0.01 \times 0.99 = 0.01281$. The mean aggregate claim is therefore $13n$ and the variance of the aggregate claims is $12810n$. The total premium is $13.2n$. The probability that the total claims exceed total premiums is therefore $1 - \Phi\left(\frac{13.2n-13n}{\sqrt{12810n}}\right) < 0.2$. This means that $\frac{13.2n-13n}{\sqrt{12810n}} = 0.00176707682335\sqrt{n} > 0.8416212$ This means $n > \left(\frac{0.8416212}{0.00176707682335}\right)^2 = 226841.479733$.

So at least 226841 lives.

15 Bayesian Estimation

15.1 Bayesian Estimation

15.2 Inference and Prediction

15.3 Conjugate Priors

16 Model Selection

16.3 Graphical Comparison

23

The log-likelihood of this Pareto distribution is

$$14\left(\log(\alpha)+\alpha\log(\theta)\right)-(\alpha+1)\left(\log(\theta+325)+\log(\theta+692)+\log(\theta+1340)+\log(\theta+1784)+\log(\theta+1920)+\log(\theta+2503)+\log(\theta+\right.$$

Differentiating with respect to α and θ give

$$\frac{14}{\alpha}=\left(\log(\theta+325)+\log(\theta+692)+\log(\theta+1340)+\log(\theta+1784)+\log(\theta+1920)+\log(\theta+2503)+\log(\theta+3238)+\log(\theta+4054)+\right.$$

$$\frac{\alpha}{\theta}=(\alpha+1)\left(\frac{1}{\theta+325}+\frac{1}{\theta+692}+\frac{1}{\theta+1340}+\frac{1}{\theta+1784}+\frac{1}{\theta+1920}+\frac{1}{\theta+2503}+\frac{1}{\theta+3238}+\frac{1}{\theta+4054}+\frac{1}{\theta+5862}+\frac{1}{\theta+6304}+\right.$$

$\theta=4156615$ $\alpha=934.25$

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See next slide.

29

$1 - e^{-\frac{10}{5.609949}} = 0.8317909$

30

$1 - e^{-\frac{3}{5.609949}} = 0.4141926$

31
$0.6^{0.8725} = 0.6403788$

16.4 Hypothesis Tests

32

(a) $1-\left(\frac{4156615}{4156615+5862}\right)^{934.25}=0.7319624$ $1-\left(\frac{4156615}{4156615+9984}\right)^{934.25}=0.8936835$
$D=0.1605338$

- At the 95% level, the critical value is $\frac{1.36}{\sqrt{14}}=0.3634753$.
- At the 95% level, the critical value is $\frac{1.22}{\sqrt{14}}=0.3260587$.

so we cannot reject the model.
(b) We have that $F(x)=1-\frac{\theta^\alpha}{(x+\theta)^\alpha}$, so the statistic is

$$n\int_t^u \frac{\left(F_n(x)+\frac{\theta^\alpha}{(x+\theta)^\alpha}-1\right)^2}{\left(\frac{\theta^\alpha((x+\theta)^\alpha-\theta^\alpha)}{(x+\theta)^{2\alpha}}\right)}\left(\frac{\alpha\theta^\alpha}{(x+\theta)^{\alpha+1}}\right)dx$$

$$n\int_t^u \alpha(x+\theta)^{\alpha-1}\frac{\left(F_n(x)+\frac{\theta^\alpha}{(x+\theta)^\alpha}-1\right)^2}{((x+\theta)^\alpha-\theta^\alpha)}dx$$

$$n\int_t^u \alpha\frac{(F_n(x)(x+\theta)^\alpha-((x+\theta)^\alpha-\theta^\alpha))^2}{(x+\theta)^{\alpha+1}((x+\theta)^\alpha-\theta^\alpha)}dx$$

$$n\int_t^u \frac{\alpha}{(x+\theta)}\left(F_n(x)^2\frac{(x+\theta)^\alpha}{((x+\theta)^\alpha-\theta^\alpha)}-2F_n(x)+\frac{((x+\theta)^\alpha-\theta^\alpha)}{(x+\theta)^\alpha}\right)dx$$

For a constant value $F_n(x)=c$, we have

$$\begin{aligned}
&\int_a^b c^2\frac{\alpha(x+\theta)^{\alpha-1}}{((x+\theta)^\alpha-\theta^\alpha)}-\frac{2\alpha c}{x+\theta}+\frac{\alpha((x+\theta)^\alpha-\theta^\alpha)}{(x+\theta)^{\alpha+1}}dx\\
&=c^2\left[\log((x+\theta)^\alpha-\theta^\alpha)\right]_a^b-2\alpha c\left[\log(x+\theta)\right]_a^b+\alpha\left[\log(x+\theta)\right]_a^b-\left[-\frac{\theta^\alpha}{(x+\theta)^\alpha}\right]_a^b\\
&=c^2\log\left(\frac{(b+\theta)^\alpha-\theta^\alpha}{(a+\theta)^\alpha-\theta^\alpha}\right)+\alpha(1-2c)\log\left(\frac{b+\theta}{a+\theta}\right)+\frac{\theta^\alpha}{(b+\theta)^\alpha}-\frac{\theta^\alpha}{(a+\theta)^\alpha}\\
&=c^2\log\left(\frac{(a+\theta)^\alpha((b+\theta)^\alpha-\theta^\alpha)}{(b+\theta)^\alpha((a+\theta)^\alpha-\theta^\alpha)}\right)+\alpha(1-c)^2\log\left(\frac{b+\theta}{a+\theta}\right)+\frac{\theta^\alpha}{(b+\theta)^\alpha}-\frac{\theta^\alpha}{(a+\theta)^\alpha}
\end{aligned}$$

For our example, if we let $t=0$ and $u=\infty$, we have the following:

$$
\begin{aligned}
&14\left(\alpha\left(1^2\log\left(\frac{325+\theta}{\theta}\right)+\left(\frac{13}{14}\right)^2\log\left(\frac{692+\theta}{325+\theta}\right)+\left(\frac{12}{14}\right)^2\log\left(\frac{1340+\theta}{692+\theta}\right)+\left(\frac{11}{14}\right)^2\log\left(\frac{1784+\theta}{1340+\theta}\right)\right.\right.\\
&+\left(\frac{10}{14}\right)^2\log\left(\frac{1920+\theta}{1784+\theta}\right)+\left(\frac{9}{14}\right)^2\log\left(\frac{2503+\theta}{1920+\theta}\right)+\left(\frac{8}{14}\right)^2\log\left(\frac{3238+\theta}{2503+\theta}\right)+\left(\frac{7}{14}\right)^2\log\left(\frac{4054+\theta}{3238+\theta}\right)\\
&+\left(\frac{6}{14}\right)^2\log\left(\frac{5862+\theta}{4054+\theta}\right)+\left(\frac{6}{14}\right)^2\log\left(\frac{6304+\theta}{5862+\theta}\right)+\left(\frac{5}{14}\right)^2\log\left(\frac{6926+\theta}{6304+\theta}\right)+\left(\frac{4}{14}\right)^2\log\left(\frac{6926+\theta}{6304+\theta}\right)\\
&\left.+\left(\frac{3}{14}\right)^2\log\left(\frac{8120+\theta}{6926+\theta}\right)+\left(\frac{2}{14}\right)^2\log\left(\frac{9176+\theta}{8120+\theta}\right)+\left(\frac{1}{14}\right)^2\log\left(\frac{9984+\theta}{9176+\theta}\right)\right)\\
&\left.+\left(\frac{1}{14}\right)^2\log\left(\frac{1-\left(\frac{\theta}{692+\theta}\right)^\alpha}{1-\left(\frac{\theta}{325+\theta}\right)^\alpha}\right)+\cdots+\left(\frac{14}{14}\right)^2\log\left(\frac{1}{1-\left(\frac{\theta}{9984+\theta}\right)^\alpha}\right)-1\right)\\
&=0.3873562
\end{aligned}
$$

So the model cannot be rejected.

34
If the parameter of the Exponential distribution is θ, then the log-likelihood of the data is

$$742\log(1-e^{-\left(\frac{5000}{\theta}\right)})+1304\log(e^{-\left(\frac{5000}{\theta}\right)}-e^{-\left(\frac{10000}{\theta}\right)})+1022\log(e^{-\left(\frac{10000}{\theta}\right)}-e^{-\left(\frac{15000}{\theta}\right)})+$$
$$830\log(e^{-\left(\frac{15000}{\theta}\right)}-e^{-\left(\frac{20000}{\theta}\right)})+211\log(e^{-\left(\frac{20000}{\theta}\right)}-e^{-\left(\frac{25000}{\theta}\right)})-143\left(\frac{25000}{\theta}\right)$$

Taking the derivative with respect to θ, we get

$$742\frac{5000e^{-\left(\frac{5000}{\theta}\right)}}{\theta^2(1-e^{-\left(\frac{5000}{\theta}\right)})}+1304\frac{(5000e^{-\left(\frac{5000}{\theta}\right)}-10000e^{-\left(\frac{10000}{\theta}\right)})}{\theta^2(e^{-\left(\frac{5000}{\theta}\right)}-e^{-\left(\frac{10000}{\theta}\right)})}+$$
$$1022\frac{(10000e^{-\left(\frac{10000}{\theta}\right)}-15000e^{-\left(\frac{15000}{\theta}\right)})}{\theta^2(e^{-\left(\frac{10000}{\theta}\right)}-e^{-\left(\frac{15000}{\theta}\right)})}+830\frac{(15000e^{-\left(\frac{10000}{\theta}\right)}-20000e^{-\left(\frac{20000}{\theta}\right)})}{\theta^2(e^{-\left(\frac{15000}{\theta}\right)}-e^{-\left(\frac{20000}{\theta}\right)})}+$$
$$211\frac{(20000e^{-\left(\frac{20000}{\theta}\right)}-25000e^{-\left(\frac{25000}{\theta}\right)})}{\theta^2(e^{-\left(\frac{20000}{\theta}\right)}-e^{-\left(\frac{25000}{\theta}\right)})}-143\frac{25000}{\theta^2}=0$$

Multiplying by $\frac{\theta^2(1-e^{-\frac{5000}{\theta}})}{5000}$ gives

$$742e^{-\left(\frac{5000}{\theta}\right)}+1304(1-2e^{-\left(\frac{5000}{\theta}\right)})+1022(2-3e^{-\left(\frac{5000}{\theta}\right)})+830(3-4e^{-\left(\frac{5000}{\theta}\right)})+$$
$$211(4-5e^{-\left(\frac{5000}{\theta}\right)})-143(5-5e^{-\left(\frac{5000}{\theta}\right)})=0$$
$$5967-10076e^{-\left(\frac{5000}{\theta}\right)}=0$$
$$e^{-\left(\frac{5000}{\theta}\right)}=\frac{5967}{10076}$$
$$\theta=\frac{5000}{\log\left(\frac{10076}{5967}\right)}$$
$$=9543.586$$

This gives the following table

Claim Amount	O_i	E_i	$\frac{(O_i-E_i)^2}{E_i}$
0–5,000	742	1733.969	567.49
5,000–10,000	1304	1026.855	74.80
10,000–15,000	1022	608.103	281.71
15,000–20,000	830	360.118	613.10
20,000–25,000	211	213.262	0.02
More than 25,000	143	309.694	89.72
total			1626.85

This should be compared to a Chi-square with 5 degrees of freedom, so the model is rejected at all significance levels.

35

For the exponential distribution, the log-likelihood is

$$-\left(\frac{382}{\theta}+\frac{596}{\theta}+\frac{920}{\theta}+\frac{1241}{\theta}+\frac{1358}{\theta}+\frac{1822}{\theta}+\frac{2010}{\theta}+\frac{2417}{\theta}+\frac{2773}{\theta}+\frac{3002}{\theta}+\frac{3631}{\theta}+\frac{4120}{\theta}+\frac{4692}{\theta}+\frac{5123}{\theta}+14\log(\theta)\right)$$

This is maximised by

$$\theta=\frac{382+596+920+1241+1358+1822+2010+2417+2773+3002+3631+4120+4692+5123}{14}=2434.786$$

Which gives a log-likelihood of $-(14+14\log(2434.786))=-123.1666$.
For the Weibull distribution, the log-likelihood is

$$14\log(\tau)+(\tau-1)(\log(382)+\cdots+\log(5213))-\left(\left(\frac{382}{\theta}\right)^\tau+\cdots+\left(\frac{5123}{\theta}\right)^\tau+14\tau\log(\theta)\right)$$

Setting the derivatives with respect to θ and τ equal to zero gives:

$$\begin{aligned}
\tau\left(\frac{382^\tau}{\theta^{\tau+1}}+\cdots+\frac{5123^\tau}{\theta^{\tau+1}}-\frac{14}{\theta}\right)&=0\\
\frac{382^\tau+\cdots+5123^\tau}{14}&=\theta^\tau\\
\frac{14}{\tau}+(\log(382)+\cdots+\log(5213))-\left(\left(\frac{382}{\theta}\right)^\tau\log\left(\frac{382}{\theta}\right)+\cdots+\left(\frac{5123}{\theta}\right)^\tau\log\left(\frac{5123}{\theta}\right)\right)-14\log(\theta)&=0\\
\frac{14}{\tau}+\left(1-\left(\frac{382}{\theta}\right)^\tau\right)\log(382)+\cdots+\left(1-\left(\frac{5123}{\theta}\right)^\tau\right)\log(5123)&=0
\end{aligned}$$

This gives the solution $\tau=1.695356$ and $\theta=2729.417$
$l(x;\tau,\theta)=-120.7921$
The log-likelihood ratio statistic is therefore

$$2(-120.7921-(-123.1666))=4.749$$

For a Chi-square with 1 degree of freedom, this has a p-value 0.04703955, so the Weibull model is preferred at the 5% significance level.

Study Note: Information Criteria

36

For the inverse exponential distribution, with parameter θ the likelihood is $\prod_{i=1}^{n} \frac{\theta e^{-\frac{\theta}{x}}}{x^2}$ and the log-likelihood is

$$l(\theta) = n\log(\theta) + \sum_{i=1}^{n} -2\log(x) - \frac{\theta}{x}$$

$$\frac{dl(\theta)}{d\theta} = \frac{n}{\theta} - \sum_{i=1}^{n} \frac{1}{x}$$

So the likelihood is maximised by $\theta = \frac{n}{\sum_{i=1}^{n} \frac{1}{x}} = 1399.291$. This gives a log-likelihood of $14\log(1399.291) - \sum_{i=1}^{n} 2\log(x) - \theta \sum_{i=1}^{n} \frac{1}{x} = -124.292$.

Now for the AIC, we get:

Weibull: $-120.7921 - 2 = -122.7921$ Inverse Exponential: $-124.292 - 1 = -125.292$

For BIC, we get

Weibull: $-120.7921 - \frac{2}{2}\log(14) = - - 123.4312$ Inverse Exponential: $-124.292 - \frac{1}{2}\log(14) = -125.6115$

IRLRPCI 2 Coverages

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Advantages of tort system	Advantages of no-fault system
Increases coverage costs for at-fault drivers, thus increasing the incentive to drive carefully.	Reduces litigation costs
More flexibility to taylor payments to injured parties needs. Under no-fault system, benefits usually defined by a formula.	Evidence shows that under tort system, small claims are overcompensated, whereas larger claims are undercompensated.

38 80% of the house price is $0.8 \times 350000 = \$280,000$, so the coinsurance pays $\frac{260000}{280000}$ of the loss, which is $\frac{260000 \times 70000}{280000} = \$65,000$.

39 (a) For a Pareto distribution, the probability that a loss exceeds x is $\left(1+\frac{x}{\theta}\right)^{-\alpha}$, so in this case, for any particular loss, it is 101^{-2}. Each policy has a 0.5 probability of producing a loss, and therefore a $\frac{1}{2\times 101^2}$ probability of producing a loss exceeding \$1,000,000. This means that the probability of a single loss exceeding \$1,000,000 is $1-\left(1-\frac{1}{2\times 101^2}\right)^{100}=0.004889607$.

(b) Calculating the exact probability is difficult.

For the Pareto distribution, the expectation of the limited loss random variable is

$$\begin{aligned}
\mathbb{E}(X\wedge 100000) &= \int_0^{100000}\frac{1}{\left(1+\frac{x}{10000}\right)^2}\,dx\\
&= 10000\int_0^{10}\frac{1}{(1+u)^2}\,du\\
&= 10000\int_1^{11}a^{-2}\,da\\
&= 10000\left[-a^{-1}\right]_1^{11}\\
&= 10000\left(1-\frac{1}{11}\right)\\
&= \frac{100000}{11}
\end{aligned}$$

We also calculate

$$\begin{aligned}
\mathbb{E}((X\wedge 100000)^2) &= \int_0^{100000}\frac{2x}{\left(1+\frac{x}{10000}\right)^2}\,dx\\
&= 10000^2\int_0^{10}\frac{2u}{(1+u)^2}\,du\\
&= 10000^2\int_1^{11}2(a-1)a^{-2}\,da\\
&= 10000^2\int_1^{11}2a^{-1}-2a^{-2}\,da\\
&= 2\times 10^8\left[\log(a)+a^{-1}\right]_1^{11}\\
&= 2\times 10^8\left(\log(11)-\frac{10}{11}\right)
\end{aligned}$$

This gives us

$$\begin{aligned}
\mathrm{Var}(X\wedge 100000) &= 10^8\left(2\log(11)-\frac{20}{11}-\frac{100}{121}\right)\\
&= 21,5116,245
\end{aligned}$$

Now the losses per policy are either X or 0 with probability 0.5. The expected loss per policy is therefore $\frac{50000}{11}$, while the variance of loss per policy is $\left(\frac{50000}{11}\right)^2 \times \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times 215116245 = 112,723,412$ The expected aggregate loss is therefore $\frac{5000000}{11} = \$454,545$ and the variance of aggregate loss is 11,272,341,200. The standard deviation is therefore \$106,171. The loss of \$1,000,000 is therefore 5.137496 standard deviations above the mean. If we use a normal approximation, the probability of aggregate losses exceeding \$1,000,000 is 1.392117×10^{-7}.

It is unclear how good the normal approximation is, so I also simulated 1,000,000 random aggregate losses, and found that in 82 of them, the aggregate loss exceeded \$1,000,000. Clearly, the normal approximation underestimates this probability, but it is still far less than the probability in (a).

IRLRPCI 4 Loss Reserving

4.6 Loss Reserving Methods

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Claim Type	Policy Year	Earned Premiums	Expected Loss Ratio	Expected Claims	Losses paid to date	Reserves needed
Collision	2014	\$200,000	0.79	\$158,000	\$130,000	\$28,000
	2015	\$250,000	0.79	\$197,500	\$110,000	\$87,500
	2016	\$270,000	0.77	\$207,900	\$60,000	\$147,900
Comprehensive	2014	\$50,000	0.74	\$37,000	\$36,600	\$400
	2015	\$60,000	0.72	\$43,200	\$44,300	\$0
	2016	\$65,000	0.75	\$48,750	\$41,400	\$7,350
Bodily Injury	2014	\$300,000	0.73	\$219,000	\$86,000	\$133,000
	2015	\$500,000	0.73	\$365,000	\$85,000	\$280,000
	2016	\$600,000	0.72	\$432,000	\$12,000	\$420,000

The loss reserves needed are therefore $28000+87500+147900+400+0+7350+133000+280000+420000 = \$1,104,150$.

41

The average, 3-year average and mean loss development factors are:

Accident year	Development year 1/0	2/1	3/2	4/3	5/4
Average	1.187962	1.200218	1.11665	1.052196	1.010355
3-year average	1.143815	1.117381	1.11665	1.052196	1.010355
Mean	$\frac{47549}{40254} = 1.181224$	$\frac{43782}{37279} = 1.174441$	$\frac{36398}{32559} = 1.117909$	$\frac{23967}{22966} = 1.043586$	$\frac{11709}{11589} = 1.010355$

The estimated future cumulative payments are then calculated by multiplying the most recent cumulative payment by the corresponding loss development factors. The three methods result in the following estimated cumulative payments:

Average:

Accident year	Development year 0	1	2	3	4	5
2012					12378	12506
2013				13432	14133	14279
2014			11223	12532	13186	13323
2015		10270	12326	13764	14483	14632
2016	11290	13412	16097	17975	18913	19109

3-year average:

Accident year	Development year 0	1	2	3	4	5
2012					12378	12506
2013				13432	14133	14279
2014			11223	12532	13186	13323
2015		10270	11476	12814	13483	13623
2016	11290	12914	14429	16113	16954	17129

mean:

Accident year	Development year 0	1	2	3	4	5
2012					12378	12506
2013				13432	14017	14163
2014			11223	12546	13093	13229
2015		10270	12062	13484	14071	14217
2016	11290	13336	15662	17509	18272	18461

42

First we calculate the expected Loss payments. Using the loss development factors, the proportion of payments made in each year is:

Cumulative	0.5889500	0.6996502	0.8397328	0.9376876	0.9866311	1
Proportion	0.5889500	0.1107002	0.1400826	0.0979548	0.0489435	0.0133689

This leads to expected payments:

Policy	Expected	Development Year					
Year	loss	0	1	2	3	4	5
2012	129,600	76,327.92	14,346.75	18,154.70	12,694.95	6,343.08	1,732.60
2013	147,600	86,929.02	16,339.35	20,676.19	14,458.13	7,224.07	1,973.24
2014	151,200	89,049.24	16,737.87	21,180.48	14,810.77	7,400.26	2,021.37
2015	158,400	93,289.68	17,534.91	22,189.08	15,516.04	7,752.66	2,117.63
2016	194,400	114,491.88	21,520.12	27,232.05	19,042.42	9,514.62	2,598.90

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Total reserve: \$13,403.51

17 Introduction and Limited Fluctuation Credibility

17.2 Limited Fluctuation Credibility Theory

17.3 Full Credibility

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(a) The number of claims made is a binomial distribution with $n = 372 \times 7 = 2604$ and some unknown p. The expected number of claims is np and the variance is $np(1-p)$, so the relative error $\frac{\overline{X}-\xi}{\xi}$ is approximately normally distributed with mean zero and variance $\frac{1-p}{np}$. We therefore want to check whether $\Phi\left(\frac{0.05}{\sqrt{\frac{1-p}{np}}}\right) \geqslant 0.975$ (two-sided confidence interval).

In this example, the total number of claims in seven years of experience is 9. This sets $p = \frac{9}{2604}$, and

$$\Phi\left(\frac{0.05}{\sqrt{\frac{1-p}{np}}}\right) = \Phi\left(\frac{0.15}{\sqrt{1-\frac{9}{2604}}}\right) = 0.5597202 < 0.975$$

So the company should not assign full credibility.

(b) Suppose we continue with the assumption that $p = \frac{9}{2604}$. Then we want to find the n such that

$$\Phi\left(\frac{0.05}{\sqrt{\frac{1-p}{np}}}\right) = \Phi\left(\frac{0.15\sqrt{n}}{\sqrt{2595}}\right) = 0.975$$

$$\frac{0.15\sqrt{n}}{\sqrt{2595}} = 1.96$$

$$n = \frac{1.96^2 \times 2595}{0.15^2} = 443064.5$$

If the company continues to employ 372 employees, then this equates to 1191.034 years.

47

Recall that we had

$$\begin{aligned}
\Phi\left(\frac{0.05}{\sqrt{\frac{1-p}{np}}}\right) &= 0.975 \\
0.05\sqrt{\frac{np}{1-p}} &= 1.96 \\
\frac{np}{1-p} &= 39.2^2 \\
np &= 1536.64(1-p) \\
(n+1536.64)p &= 1536.64 \\
p &= \frac{1536.64}{n+1536.64} \\
np &= \frac{1536.64n}{n+1536.64}
\end{aligned}$$

If p is small (and n is large), we can approximate $1 - p = 0$, so the standard for full credibility is 1568.64 claims. If n is smaller, then the standard for full credibility also gets smaller. For example, if $n = 1536.64$, then the standard for full credibility is only half as much.

48

(a)

Based on the data, the coefficient of variation is $\frac{3605.52}{962.14} = 3.747396$. Assuming the number of claims is large enough to use a normal approximation, we have that the critical value is 1.96 at the 95% confidence level. This means that the coefficient of variation for the average $\overline{X}$ is $\frac{3.747396}{\sqrt{41876}} = 0.01831247$. Multiplying by 1.96 gives us the relative 95% confidence interval as 0.03589244. Since this is less than 0.05, the company should assign full credibility to this data.

(b) The insurance company will assign full credibility if

$$\frac{3.747396}{\sqrt{n}} \times 1.96 \leqslant 0.05$$

$$n \geqslant \left(\frac{1.96 \times 3.747396}{0.05}\right)^2 = 21579$$

17.4 Partial Credibility

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The partial credibility assigned is $Z = \sqrt{\frac{7}{1191.034}} = 0.0766632$
The credibility premium is therefore

$$0.0766632 \times \frac{126000}{372} + 0.9233368 \times 1000 = \$949.303367742$$

Using 1568.64 claims as the standard for full credibility gives $Z = \sqrt{\frac{9}{1568.64}} = 0.075746$
The credibility premium is therefore

$$0.075746 \times 338.7097 + 0.924254 \times 1000 = \$949.91$$

50

(a) The credibility for claim frequency is $Z = \sqrt{\frac{19}{421}} = 0.2124397$, so the credibility estimate for claim frequency is $0.2124397 \times 1.9 + 0.7875603 \times 1.2 = 1.348708$.

The credibility for claim severity is $Z = \sqrt{\frac{19}{1240}} = 0.1237844$, so the credibility estimate for claim severity is $0.1237844 \times \frac{5822}{19} + 0.8762156 \times 230 = 239.4597$. The credibility estimate for aggregate claims is therefore $1.348708 \times 239.4597 = \$322.9613$.

(b) The credibility for claim frequency is $Z = \sqrt{\frac{19}{1146}} = 0.128761$, so the credibility estimate for claim frequency is $0.128761 \times 1.9 + 0.871239 \times 1.2 = 1.290133$.

The credibility for claim severity is $Z = \sqrt{\frac{19}{611}} = 0.1763422$, so the credibility estimate for claim severity is $0.1763422 \times \frac{5822}{19} + 0.8236578 \times 230 = 243.4763$. The credibility estimate for aggregate claims is therefore $1.290133 \times 243.4763 = \$314.1168$.

(c) The credibility for aggregate losses is $Z = \sqrt{\frac{10}{400}} = 0.1581139$. The credibility premium is therefore $0.1581139 \times 582.2 + 0.8418861 \times 276 = \$324.4145$.

(d) The credibility for aggregate losses is $Z = \sqrt{\frac{10}{1000}} = 0.1$. The credibility premium is therefore $0.1 \times 582.2 + 0.9 \times 276 = \$306.62$.

17.5 Problems with this Approach

51

Using a normal approximation, the standard for full credibility is

$$\Phi\left(\frac{r\sqrt{n}}{\tau}\right) \geqslant 1 - \frac{p}{2}$$

where τ is the coefficient of variation of X. For our data, we have

$$\tau = \frac{\sqrt{8240268} \times 3722}{3506608} = 3.046911$$

The standard for full credibility is therefore given by

$$\sqrt{n} = \frac{3.046911}{r}\left(\Phi^{-1}\left(1 - \frac{p}{2}\right)\right)$$

The credibility is

$$Z = \sqrt{\frac{3722}{n}} = \frac{\sqrt{3722}}{3.046911\Phi^{-1}\left(1 - \frac{p}{2}\right)}r = \frac{20.02297r}{\Phi^{-1}\left(1 - \frac{p}{2}\right)}$$

18 Greatest Accuracy Credibility

18.2 Conditional Distributions and Expectation

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(a) Let $\Theta = 1$ for frequent drivers, and $\Theta = 0$ for infrequent drivers. Then

$$\begin{aligned}\mathbb{E}(X|\Theta = 1) &= 0.4\\ \mathbb{E}(X|\Theta = 0) &= 0.1\\ \mathrm{Var}(X|\Theta = 1) &= 0.4\\ \mathrm{Var}(X|\Theta = 0) &= 0.1\end{aligned}$$

so

$$\mathbb{E}(X) = \mathbb{E}(\mathbb{E}(X|\Theta) = 0.75 \times 0.4 + 0.25 \times 0.1 = 0.325$$

and

$$\mathrm{Var}(X) = \mathbb{E}(\mathrm{Var}(X|\theta)) + \mathrm{Var}(\mathbb{E}(X|\Theta)) = 0.325 + 0.3^2 \times 0.25 \times 0.75 = 0.325 + 0.016875 = 0.341875$$

(b)

$$P(X = 0|\Theta) = \begin{cases} e^{-0.4} & \text{if } \Theta = 1 \\ e^{-0.1} & \text{if } \Theta = 0 \end{cases}$$

So

$$P(\Theta = 1|X = 0) = \frac{0.75e^{-0.4}}{0.75e^{-0.4} + 0.25e^{-0.1}} = 0.6896776$$

Therefore the new expectation and variance are:

$$\mathbb{E}(X) = \mathbb{E}(\mathbb{E}(X|\Theta) = 0.6896776 \times 0.4 + 0.3103224 \times 0.1 = 0.3069033$$

and

$$\mathrm{Var}(X) = \mathbb{E}(\mathrm{Var}(X|\theta)) + \mathrm{Var}(\mathbb{E}(X|\Theta)) = 0.3069033 + 0.3^2 \times 0.3103224 \times 0.6896776 = 0.325 + 0.016875 = 0.3261653$$

18.3 Bayesian Methodology

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(a) We have $\mathbb{E}(X|\Theta=\theta)=\frac{\theta}{2}$, so

$$\mathbb{E}(X)=\mathbb{E}(\mathbb{E}(X|\Theta=\theta))=\mathbb{E}\left(\frac{\Theta}{2}\right)=150$$

(b) The joint density function is

$$f_{X,\Theta}(x,\theta)=\left(\frac{\theta^2}{2\times 100^3}e^{-\frac{\theta}{100}}\right)\left(\frac{\theta^3}{2x^4}e^{-\frac{\theta}{x_1}}\right)$$

For samples, x_1 and x_2, the joint density is therefore

$$\begin{aligned}\left(\frac{\theta^2}{2000000}e^{-\frac{\theta}{100}}\right)\left(\frac{\theta^3}{2{x_1}^4}e^{-\frac{\theta}{x_1}}\right)\left(\frac{\theta^3}{2{x_2}^4}e^{-\frac{\theta}{x_2}}\right)\\=\frac{\theta^8}{8000000{x_1}^4{x_2}^4}e^{-\theta\left(\frac{1}{100}+\frac{1}{x_1}+\frac{1}{x_2}\right)}\end{aligned}$$

The posterior distribution of Θ is therefore a gamma distribution with $\alpha=9$ and $\theta=\frac{1}{\frac{1}{100}+\frac{1}{x_1}+\frac{1}{x_2}}=$ 43.29897.

The expected aggregate losses are given by

$$\begin{aligned}\mathbb{E}(X)&=\mathbb{E}(\mathbb{E}(X|\Theta))=\frac{\mathbb{E}(\Theta)}{2}\\&=4.5\times 43.29897\\&=194.845365\end{aligned}$$

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(a) We have

$$\mathbb{E}(X) = \mathbb{E}(\mathbb{E}(X|\Lambda)) = \mathbb{E}(\Lambda) = 1$$

(b) The posterior distribution is a Gamma distribution with $\alpha = 0.5 + m$ and $\theta = \frac{2}{1+2n}$. We therefore have

$$\begin{aligned}
\mathbb{E}(X) = \mathbb{E}(\mathbb{E}(X|\Lambda)) = \mathbb{E}(\Lambda) \\
&= \frac{2(m+0.5)}{1+2n} \\
&= \frac{2m+1}{2n+1} \\
&= \left(\frac{2n}{1+2n}\right)\left(\frac{m}{n}\right) + \left(\frac{1}{1+2n}\right)
\end{aligned}$$

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The posterior density function is proportional to

$$4\lambda^m e^{-n\lambda}\frac{3^4}{(3+\lambda)^5}$$

We have that the posterior expected number of claims is the posterior expected value of Λ, which is given by

$$\frac{\int_0^\infty \frac{\lambda^{m+1}e^{-n\lambda}}{(3+\lambda)^5}\,d\lambda}{\int_0^\infty \frac{\lambda^{m}e^{-n\lambda}}{(3+\lambda)^5}\,d\lambda}$$

Substituting $u=\lambda+3$, these integrals become

$$\frac{\int_3^\infty u^{-5}(u-3)^{m+1}e^{-nu}\,du}{\int_3^\infty u^{-5}(u-3)^{m}e^{-nu}\,du}$$

The credibility estimate is then given in the following table:

	1	2	3	4	5	6	7	8	9	10
0	0.4331	0.2937	0.2243	0.1821	0.1534	0.1327	0.1170	0.1046	0.0946	0.0864
1	0.9261	0.6073	0.4580	0.3693	0.3101	0.2675	0.2354	0.2103	0.1900	0.1734
2	1.4785	0.9396	0.7003	0.5614	0.4697	0.4044	0.3552	0.3169	0.2862	0.2609
3	2.0874	1.2891	0.9506	0.7579	0.6321	0.5430	0.4764	0.4246	0.3831	0.3491
4	2.7487	1.6543	1.2081	0.9584	0.7970	0.6833	0.5987	0.5331	0.4806	0.4377
5	3.4571	2.0336	1.4722	1.1627	0.9642	0.8252	0.7221	0.6424	0.5788	0.5269
6	4.2067	2.4256	1.7423	1.3704	1.1336	0.9686	0.8466	0.7525	0.6776	0.6165
7	4.9919	2.8288	2.0178	1.5811	1.3050	1.1134	0.9721	0.8633	0.7769	0.7065
8	5.8073	3.2420	2.2981	1.7948	1.4782	1.2594	1.0985	0.9749	0.8768	0.7970
9	6.6477	3.6640	2.5829	2.0110	1.6531	1.4065	1.2257	1.0870	0.9771	0.8878

We compare this to the table of $\frac{2m+1}{2n+1}$ that we get from the Gamma prior.

	1	2	3	4	5	6	7	8	9	10
0	0.3333	0.2000	0.1429	0.1111	0.0909	0.0769	0.0667	0.0588	0.0526	0.0476
1	1.0000	0.6000	0.4286	0.3333	0.2727	0.2308	0.2000	0.1765	0.1579	0.1429
2	1.6667	1.0000	0.7143	0.5556	0.4545	0.3846	0.3333	0.2941	0.2632	0.2381
3	2.3333	1.4000	1.0000	0.7778	0.6364	0.5385	0.4667	0.4118	0.3684	0.3333
4	3.0000	1.8000	1.2857	1.0000	0.8182	0.6923	0.6000	0.5294	0.4737	0.4286
5	3.6667	2.2000	1.5714	1.2222	1.0000	0.8462	0.7333	0.6471	0.5789	0.5238
6	4.3333	2.6000	1.8571	1.4444	1.1818	1.0000	0.8667	0.7647	0.6842	0.6190
7	5.0000	3.0000	2.1429	1.6667	1.3636	1.1538	1.0000	0.8824	0.7895	0.7143
8	5.6667	3.4000	2.4286	1.8889	1.5455	1.3077	1.1333	1.0000	0.8947	0.8095
9	6.3333	3.8000	2.7143	2.1111	1.7273	1.4615	1.2667	1.1176	1.0000	0.9048

18.4 The Credibility Premium

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We are trying to choose α_i to minimise

$$\mathbb{E}\left(\mu(\Theta)-\left(\alpha_0+\sum_{i=1}^{n}\alpha_iX_i\right)\right)^2=\mathbb{E}\left(\mu(\Theta)^2-2\mu(\Theta)\left(\alpha_0+\sum_{i=1}^{n}\alpha_iX_i\right)+\left(\alpha_0+\sum_{i=1}^{n}\alpha_iX_i\right)^2\right)$$

$$=\mathbb{E}\left(\mu(\Theta)^2\right)-2\alpha_0\mathbb{E}\mu(\Theta)-\mathbb{E}\left(\mu(\Theta)\left(\sum_{i=1}^{n}\alpha_iX_i\right)\right)+{\alpha_0}^2+2\alpha_0\mathbb{E}\left(\sum_{i=1}^{n}\alpha_iX_i\right)+\mathbb{E}\left(\left(\sum_{i=1}^{n}\alpha_iX_i\right)^2\right)$$

$$=\mu^2+v^2-2\alpha_0\mu+{\alpha_0}^2+2\alpha_0\mathbb{E}\left(\sum_{i=1}^{n}\alpha_iX_i\right)-\mathbb{E}\left(\mu(\Theta)\left(\sum_{i=1}^{n}\alpha_iX_i\right)\right)+\mathbb{E}\left(\left(\sum_{i=1}^{n}\alpha_iX_i\right)^2\right)$$

Setting the derivative with respect to α_0 equal to zero yields

$$2\left(\alpha_0+\mathbb{E}\left(\sum_{i=1}^{n}\alpha_iX_i\right)-\mu\right)=0$$

That is, α_0 should be chosen to make the estimate unbiassed. Now we differentiate with respect to α_j, and set the derivative equal to zero:

$$2\left(\alpha_0\mathbb{E}(X_j)-\mathbb{E}(\mu(\Theta)X_j)+\mathbb{E}\left(X_j\sum_{i=1}^{n}\alpha_iX_i\right)\right)=0$$

$$2\left(\mathbb{E}\left(\mu(\Theta)-\sum_{i=1}^{n}\alpha_iX_i\right)\mathbb{E}(X_j)-\mathbb{E}\left(X_j\left(\mu(\Theta)-\sum_{i=1}^{n}\alpha_iX_i\right)\right)\right)=0$$

$$\mathrm{Cov}\left(X_j,\mu(\Theta)-\sum_{i=1}^{n}\alpha_iX_i\right)=0$$

$$\mathrm{Cov}\left(X_j,\mu(\Theta)\right)=\sum_{i=1}^{n}\alpha_i\,\mathrm{Cov}\left(X_j,X_i\right)$$

Since X_i and X_{n+1} are conditionally independent given $\mu(\Theta)$, we have that $\mathrm{Cov}\left(X_j,\mu(\Theta)\right)=\mathrm{Cov}\left(X_j,X_{n+1}\right)$

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In this situation, the second normal equations becomes:

$$\rho = \left(\sum_{i=1}^{n} \alpha_i \rho\right) + \alpha_j \sigma^2$$
$$\alpha_j = \frac{\rho\left(1 - \sum_{i=1}^{n} \alpha_i\right)}{\sigma^2}$$

So all the α_j are equal to a common value α, and we get

$$\alpha = \frac{\rho\,(1 - n\alpha)}{\sigma^2}$$

Now we have $\mathbb{E}(X_{n+1}) = \mu = \mathbb{E}(X_i)$ The first normal equation then becomes

$$\begin{aligned} \mu &= \alpha_0 + \left(\sum_{i=1}^{n} \alpha_i\right)\mu \\ &= \alpha_0 + n\alpha\mu \\ \alpha_0 &= (1 - n\alpha)\mu \end{aligned}$$

We can therefore rewrite our credibility estimate as

$$Z\overline{X} + (1 - Z)\mu$$

where $Z = n\alpha$. We can then solve:

$$\begin{aligned} \frac{Z}{n} &= \frac{\rho\,(1 - Z)}{\sigma^2} \\ \sigma^2 Z &= n\rho(1 - Z) \\ (\sigma^2 + n\rho)Z &= n\rho \\ Z &= \frac{n}{n + \frac{\sigma^2}{\rho}} \end{aligned}$$

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Let the coefficients of the X_i be α_i, and let the coefficients of Y_i be β_i. The normal equations are:

$$\begin{aligned}\mu+\nu &= \alpha_0+\sum_{j=1}^{n}\alpha_i\mu+\sum_{k=1}^{m}\beta_i\nu\\ \rho+\xi &= \sum_{j\neq i}\alpha_j\rho+\sum_{k=1}^{m}\beta_k\xi+\alpha_i\sigma^2\\ \zeta+\xi &= \sum_{j=1}^{n}\alpha_j\xi+\sum_{k\neq i}\beta_k\zeta+\beta_i\tau^2\end{aligned}$$

From these, we deduce that $\beta_i(\tau-\zeta)=\beta_j(\tau-\zeta)$, and so $\beta_i=\beta_j=\beta$ (assuming the Y_i are not perfectly correlated). Similarly, $\alpha_i=\alpha_j=\alpha$. Substituting these into the normal equations gives:

$$\begin{aligned}\mu+\nu &= \alpha_0+n\alpha\mu+m\beta\nu\\ \rho+\xi &= \alpha((n-1)\rho+\sigma^2)+m\beta\xi\\ \zeta+\xi &= n\alpha\xi+\beta((m-1)\zeta+\tau^2)\end{aligned}$$

This gives

$$\begin{aligned}\left(\frac{(n-1)\rho+\tau}{n\xi}\right)(\zeta+\xi)-(\rho+\xi) &= \left(\frac{(n-1)\rho+\sigma^2}{n\xi}\right)\beta((m-1)\zeta+\tau^2)-m\beta\xi\\ \beta &= \frac{\left(\frac{(n-1)\rho+\sigma^2}{n\xi}\right)(\zeta+\xi)-(\rho+\xi)}{\left(\frac{(n-1)\rho+\sigma^2}{n\xi}\right)((m-1)\zeta+\tau^2)-m\xi}\\ &= \frac{\left((n-1)\rho+\sigma^2\right)(\zeta+\xi)-n\xi(\rho+\xi)}{((n-1)\rho+\sigma^2)\left((m-1)\zeta+\tau^2\right)-mn\xi^2}\\ \alpha &= \frac{((m-1)\zeta+\tau^2)-m\xi(\zeta+\xi)}{((n-1)\rho+\sigma^2)\left((m-1)\zeta+\tau^2\right)-mn\xi^2}\\ \alpha_0 &= (1-n\alpha)\mu+(1-m\beta)\nu\end{aligned}$$

18.5 The Buhlmann Model

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We have $Z = \frac{851}{851+\frac{84036}{23804}} = 0.9958687$, and $\overline{X} = \frac{121336}{851} = \$142.58$ so the credibility premium is

$$0.9958687 \times 142.58 + 0.0041313 \times 326 = 143.34$$

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We have $Z = \frac{10}{10+\frac{732403}{28822}} = 0.2823961$, and $\overline{X} = \frac{3224}{10} = 322.40$ so the credibility premium is

$$0.2823961 \times 322.40 + 0.7176039 \times 990 = \$801.47$$

18.6 The Buhlmann-Straub Model

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The weighted mean is $\frac{1000000}{1242} = 805.153$. The credibility is $Z = \frac{1242}{1242+\frac{81243100}{120384}} = 0.6479325$. The credibility premium is therefore

$$0.6479325 \times 805.153 + 0.4520675 \times 1243 = \$959.30$$

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The weighted mean is $\frac{14000}{\left(\frac{49}{12}\right)} = \$3,428.57$. The credibility is $Z = \frac{\left(\frac{49}{12}\right)}{\left(\frac{49}{12}\right)+\left(\frac{34280533}{832076}\right)} = 0.09017537$. The credibility premium is therefore

$$0.09017537 \times 3428.57 + 0.90981463 \times 600 = \$855.07$$

18.7 Exact Credibility

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The Bayes premium is the conditional expectation of X_{n+1} given $X_1, \ldots, X_n$. We are given that it is a linear function of X_i. That is

$$\mathbb{E}(X_{n+1}|X_1, \ldots, X_n) = \alpha_0 + \sum_{i=1}^{n} \alpha_i X_i$$

Now recall that

$$\begin{aligned}
\text{Cov}(X_i, X_{n+1}) &= \mathbb{E}(X_i X_{n+1}) - \mathbb{E}(X_i)\mathbb{E}(X_{n+1}) \\
&= \mathbb{E}(\mathbb{E}(X_i X_{n+1}|X_1, \ldots, X_n)) - \mathbb{E}(X_i)\mathbb{E}(\mathbb{E}(X_{n+1}|X_1, \ldots, X_n)) \\
&= \mathbb{E}(X_i \sum_{j=1}^{n} \alpha_j X_j) - \mathbb{E}(X_i)\mathbb{E}(\sum_{j=1}^{n} \alpha_j X_j) \\
&= \sum_{j=1}^{n} \alpha_j \left(\mathbb{E}(X_i X_j) - \mathbb{E}(X_i)\mathbb{E}(X_j)\right) \\
&= \sum_{j=1}^{n} \alpha_j \, \text{Cov}(X_i, X_j)
\end{aligned}$$

This means that the second normal equation is satisfied by the Bayes premium. We also have

$$\mathbb{E}(X_{n+1}) = \mathbb{E}(\mathbb{E}(X_{n+1}|X_1, \ldots, X_n)) = \mathbb{E}\left(\alpha_0 + \sum_{i=1}^{n} \alpha_i X_i\right)$$

So the first normal equation is satisfied. Thus the Bayes premium is the credibility premium. [Technically, need to show this is the only solution].

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The conjugate prior is

$$\pi(\theta) \propto h(\theta) e^{\alpha r(\theta)} q(\theta)^{-\beta}$$

We choose $h(\theta) = Cr'(\theta)$.

We now show:

Proposition 1. *The marginal mean is* $\frac{\alpha}{\beta}$.

Proof. We have that C is given by

$$C \int r'(\theta) e^{\alpha r(\theta)} q(\theta)^{-\beta} \, d\theta = 1$$

We note that

$$r'(\theta) e^{\alpha r(\theta)} = \frac{d}{d\theta} \left(\frac{e^{\alpha r(\theta)}}{\alpha} \right)$$

so integrating by parts gives

$$C \left(\left[\frac{e^{\alpha r(\theta)}}{\alpha} q(\theta)^{-\beta} \right] + \frac{\beta}{\alpha} \int e^{\alpha r(\theta)} q'(\theta) q(\theta)^{-\beta-1} \, d\theta \right) = 1$$

□

The posterior distribution is

$$\frac{\pi(\theta) e^{r(\theta) \sum X_i}}{q(\theta)^N} = Cr'(\theta) e^{\alpha r(\theta) - \beta \log(q(\theta))} e^{r(\theta) \sum X_i - N \log(q(\theta))} = \frac{Cr'(\theta) e^{r(\theta)\left(\alpha + \sum_{i=1}^n X_i\right)}}{q(\theta)^{\beta+N}}$$

Recall that the mean of a distribution from the linear-exponential family is

$$\frac{q'(\theta)}{r'(\theta) q(\theta)}$$

The posterior mean is therefore

$$\begin{aligned}
\mathbb{E}\left(\frac{q'(\Theta)}{r'(\Theta) q(\Theta)} \right) &= \int_{\theta_0}^{\theta_1} \frac{Cr'(\theta) e^{r(\theta)\left(\alpha + \sum_{i=1}^n X_i\right)}}{q(\theta)^{\beta+N}} \frac{q'(\theta)}{r'(\theta) q(\theta)} \, d\theta \\
&= C \int_{\theta_0}^{\theta_1} \frac{q'(\theta) e^{r(\theta)\left(\alpha + \sum_{i=1}^n X_i\right)}}{q(\theta)^{\beta+N+1}} \, d\theta \\
&= C \left(\left[\frac{e^{r(\theta)\left(\alpha + \sum_{i=1}^n X_i\right)}}{q(\theta)^{\beta+N}} \right]_{\theta_0}^{\theta_1} - \int_{\theta_0}^{\theta_1} \frac{r'(\theta) e^{r(\theta)\left(\alpha + \sum_{i=1}^n X_i\right)}}{q(\theta)^{\beta+N}} \, d\theta \right)
\end{aligned}$$

$$\iint \frac{h(\theta) e^{r(\theta)\left(\alpha + \sum_{i=1}^n X_i\right)}}{q(\theta)^{\beta+N}} x \frac{p(x) e^{r(\theta) x}}{q(\theta)} \, dx \, d\theta = \iint x \frac{p(x) h(\theta) e^{r(\theta)\left(\alpha + x + \sum_{i=1}^n X_i\right)}}{q(\theta)^{\beta+N+1}} \, dx \, d\theta$$

19 Empirical Bayes Parameter Estimation

19.2 Nonparametric Estimation

65

(a) The overall mean is $\frac{2770.8}{8} = 346.35$

The EPV is $\frac{60595.2+1225822.8+62760.2+192962.3+0.0+30505.0+140653.7+56385.3}{8} = \frac{1769684.5}{8} = 221210.5625$

The total variance is

$\frac{(172.80-346.35)^2+(671.60-346.35)^2+(177.80-346.35)^2+(635.40-346.35)^2+(0.00-346.35)^2+(247.00-346.35)^2+(633.60-346.35)^2+(232.60-346.35)^2}{7} = 67592.36$

The VHM is $67592.36 - \frac{221210.5625}{5} = 23350.25$

(b) The credibility of 5 years of experience is

$$Z = \frac{5}{5 + \frac{221210.5625}{23350.25}} = 0.3454569$$

The premiums are

$$\begin{aligned}
0.3454569 \times 172.80 + 0.6545431 \times 346.35 &= \$286.40 \\
0.3454569 \times 671.60 + 0.6545431 \times 346.35 &= \$458.71 \\
0.3454569 \times 177.80 + 0.6545431 \times 346.35 &= \$288.12 \\
0.3454569 \times 635.40 + 0.6545431 \times 346.35 &= \$446.20 \\
0.3454569 \times 0.00 + 0.6545431 \times 346.35 &= \$226.70 \\
0.3454569 \times 247.00 + 0.6545431 \times 346.35 &= \$312.03 \\
0.3454569 \times 633.60 + 0.6545431 \times 346.35 &= \$445.58 \\
0.3454569 \times 232.60 + 0.6545431 \times 346.35 &= \$307.05
\end{aligned}$$

66 In general, we can write the aggregate loss per unit of exposure for the ith company in the jth year as $A_{ij} = M_i + E_{ij}$ where M_i is the mean aggregate loss per unit of exposure for the ith company, and E_{ij} is the process variation, which has mean 0 and expected variance $\frac{\sigma^2}{m_{ij}}$. The estimated mean for each company is

$$A_i = \frac{1}{m_i}\sum m_{ij}A_{ij} = \frac{1}{m_i}\sum m_{ij}M_i + \frac{1}{m_i}\sum m_{ij}E_{ij} = M_i + \frac{1}{m_i}\sum m_{ij}E_{ij}$$

We let $E_i = \frac{1}{m_i}\sum m_{ij}E_{ij}$, and see that E_i has mean 0 and expected variance $\frac{\sigma^2}{m_i}$.

Now we consider

$$\sum m_i(A_i - \hat{\mu})^2 = \sum m_i\left((M_i - \overline{M_i}) + (E_i - \overline{E_i})\right)^2$$

Since the E_i are assumed to have mean 0 for each i, we should have $\operatorname{Cov}\left(M_i - \overline{M_i}, E_i - \overline{E_i}\right) = 0$. This gives

$$\mathbb{E}\left(\sum m_i(A_i - \hat{\mu})^2\right) = \sum m_i\left(\mathbb{E}((M_i - \overline{M_i})^2) + \mathbb{E}((E_i - \overline{E_i})^2)\right)$$

We have that $\frac{\sum m_i\mathbb{E}((E_i - \overline{E_i})^2)}{(n-1)} = \sigma^2$, the expected process variance. Therefore, we calculate

$$\mathbb{E}\left(\sum m_i(A_i - \hat{\mu})^2\right) - (n-1)\widehat{\sigma^2} = \sum m_i\left(\mathbb{E}((M_i - \overline{M_i})^2\right)$$

We are interested in the variance of the M_i, where the probability of each i is assumed to be $\frac{m_i}{m}$. We have that $\operatorname{Var}(M_i - \overline{M}) = \operatorname{Var}(M_i) + \operatorname{Var}(\overline{M}) - 2\operatorname{Cov}(M_i, \overline{M})$. We know that $\operatorname{Var}(M_i) = {\sigma_m}^2$ the variance of hypothetical means. Since $\overline{M} = \sum \frac{m_i}{m}M_i$, we have that $\operatorname{Var}(M) = {\sigma_m}^2\sum\frac{{m_i}^2}{m^2}$ and $\operatorname{Cov}(M_i, \overline{M}) = \frac{m_i}{m}{\sigma_m}^2$. This gives that

$$\begin{aligned}\sum m_i\operatorname{Var}(M_i - \overline{M}) &= \sum m_i{\sigma_m}^2 + m\sum\frac{{m_i}^2}{m^2}{\sigma_m}^2 - 2\sum\frac{{m_i}^2}{m}{\sigma_m}^2\\ &= {\sigma_m}^2\left(m - \sum\frac{{m_i}^2}{m}\right)\end{aligned}$$

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In total the aggregate claims were 15.7 million, and the total exposure was 14,693 lives. The average claim per life is therefore $\frac{15700000}{14693} = 1068.54$. The averages for the three companies are:

$$\frac{5300000}{3623} = 1,462.88$$
$$\frac{4000000}{4908} = \quad 815.00$$
$$\frac{6400000}{6162} = 1,038.62$$

The variances for the three companies are:

$$\frac{769(1690.51-1462.88)^2 + 928(1616.38-1462.88)^2 + 880(909.09-1462.88)^2 + 1046(1625.24-1462.88)^2}{3} = 116443575$$
$$\frac{1430(699.30-815)^2 + 1207(745.65-815)^2 + 949(632.24-815)^2 + 1322(1134.64-815)^2}{3} = 63905244$$
$$\frac{942(1167.73-1038.62)^2 + 1485(942.76-1038.62)^2 + 2031(935.50-1038.62)^2 + 1704(1173.71-1038.62)^2}{3} = 27347095$$

The expected process variance is therefore:

$$\frac{3623 \times 116443575 + 4908 \times 63905244 + 6162 \times 27347095}{14693} = 61528266$$

We have that

$$3623(1462.88-1068.54)^2 + 4908(815.00-1068.54)^2 + 6162(1038.62-1068.54)^2 = 884406185$$

We have that

$$884406185 - 2 \times 61528266 = 761349653$$

We also get

$$m - \sum \frac{{m_i}^2}{m} = 14693 - \frac{3623^2 + 4908^2 + 6162^2}{14693} = 9575.949$$

The variance of hypothetical means is $\frac{761349653}{9575.949} = 79506.44$

The credibilities of the three companies' experiences are therefore

$$Z_1 = \frac{3623}{3623 + \frac{61528266}{79506.44}} = 0.8239938$$
$$Z_2 = \frac{4908}{4908 + \frac{61528266}{79506.44}} = 0.8637989$$
$$Z_3 = \frac{6162}{6162 + \frac{61528266}{79506.44}} = 0.8884240$$

The credibility premiums per unit of exposure are therefore:

$$\begin{aligned} 0.8239938 \times 1462.88 + 0.1760062 \times 1068.54 &= \$1,393.47 \\ 0.8637989 \times 815.00 + 0.1362011 \times 1068.54 &= \quad \$849.53 \\ 0.8884240 \times 1038.62 + 0.1115760 \times 1068.54 &= \$1,041.96 \end{aligned}$$

The credibility-weighted average is

$$\frac{0.8239938 \times 1462.88 + 0.8637989 \times 815.00 + 0.8884240 \times 1038.62}{0.8239938 + 0.8637989 + 0.8884240} = \$1,099.34$$

Using this average, the credibility premiums are

$$\begin{aligned} 0.8239938 \times 1462.88 + 0.1760062 \times 1099.34 &= \$1,398.89 \\ 0.8637989 \times 815.00 + 0.1362011 \times 1099.34 &= \quad \$853.73 \\ 0.8884240 \times 1038.62 + 0.1115760 \times 1099.34 &= \$1,045.39 \end{aligned}$$

19.3 Semiparametric Estimation

68

There are a total of 3193 claims from 6210 policyholders, so the estimate for μ is $\frac{3193}{6210} = 0.5141707$. Since for a Poisson distribution the mean and variance are equal, this gives the expected process variance is also $v = 0.5141707$. We calculate the sample variance

$$\frac{6210}{6209}\left(\frac{1406 + 740 \times 4 + 97 \times 9 + 13 \times 16 + 3 \times 25}{6210} - 0.5141707^2\right) = 0.6249401$$

so the variance of hypothetical means is $0.6249401 - 0.5141707 = 0.1107694$ and the credibility of 3 years of experience is

$$Z = \frac{3}{3 + \frac{0.5141707}{0.1107694}} = 0.3925771$$

so the credibility estimate is

$$0.3925771 \times 2 + 0.6074229 \times 0.5141707 = 1.097473$$

69

42590 claims were made in 91221 years by 34285 policyholders.

The global mean is therefore $\mu = \frac{42590}{91221} = 0.4668881$ claims per year.

For the Poisson distribution, the mean is equal to the variance, so the expected process variance is also 0.4668881.

As in Question 67, the estimator for VHMs is

$$\widehat{\text{VHM}} = \frac{\sum m_i \left(\frac{n_i}{m_i} - 0.4668881\right)^2 - (n-1)\text{EPV}}{m - \frac{\sum {m_i}^2}{m}}$$

We compute

$$\begin{aligned}\sum m_i \left(\frac{n_i}{m_i} - 0.4668881\right)^2 &= 3951(0-0.4668881)^2 + 1406(1-0.4668881)^2 + 740(2-0.4668881)^2 + 97(3-0.4668881)^2 \\ &\quad + 13(4-0.4668881)^2 + 3(5-0.4668881)^2 + 2\left(3628(0-0.4668881)^2 + 2807(0.5-0.4668881)^2\right. \\ &\quad + 1023(1-0.4668881)^2 + 461(1.5-0.4668881)^2 + 104(2-0.4668881)^2 + 13(2.5-0.4668881)^2 \\ &\quad \left.+4(3-0.4668881)^2 + (4-0.4668881)^2\right) + 3\left(2967(0-0.4668881)^2\right. \\ &\quad + 4032(0.33333333-0.4668881)^2 + 2214(0.66666667-0.4668881)^2 + 890(1-0.4668881)^2 \\ &\quad + 734(1.33333333-0.4668881)^2 + 215(1.666666667-0.4668881)^2 + 131(2-0.4668881)^2 \\ &\quad \left.+22(2.33333333-0.4668881)^2 + 2(3-0.4668881)^2\right) + 4\left(1460(0-0.4668881)^2\right. \\ &\quad + 2828(0.25-0.4668881)^2 + 2204(0.5-0.4668881)^2 + 985(0.75-0.4668881)^2 \\ &\quad + 747(1-0.4668881)^2 + 358(1.25-0.4668881)^2 + 194(1.5-0.4668881)^2 + 43(1.75-0.4668881)^2 \\ &\quad \left.+8(2-0.4668881)^2\right) \\ &= 19670.9022002\end{aligned}$$

$$m - \frac{\sum {m_i}^2}{m} = 91221 - \frac{6210 \times 1^2 + 8041 \times 2^2 + 11207 \times 3^2 + 8827 \times 4^2}{91221} = 91217.92539$$

So $\widehat{\text{VHM}} = \frac{19670.9022002 - 34284 \times 0.4668881}{91217.92539} = 0.0401687559121$

Because the different policyholders have different exposure, we should use a credibility-weighted average here. We calculate the credibility for different numbers of years of history:

Years	Credibility	Total claims	Total policyholders
1	$\frac{1}{1+\frac{0.4668881}{0.0401687559121}} = 0.079219431596$	3244	6210
1	$\frac{2}{2+\frac{0.4668881}{0.0401687559121}} = 0.146808756915$	6749	8041
1	$\frac{3}{3+\frac{0.4668881}{0.0401687559121}} = 0.205153938061$	16099	11207
1	$\frac{4}{4+\frac{0.4668881}{0.0401687559121}} = 0.256030059119$	16498	8827

The credibility weighted average is therefore

$$\frac{0.079219431596 \times 3244 + 0.146808756915 \times \frac{6749}{2} + 0.205153938061 \times \frac{16099}{3} + 0.256030059119 \times \frac{16498}{4}}{0.079219431596 \times 6210 + 0.146808756915 \times 8041 + 0.205153938061 \times 11207 + 0.256030059119 \times 8827} = 0.466866294171$$

so the credibility estimate is

$$0.205153938061 \times 0.66666666667 + 0.794846061939 \times 0.466866294171 = 0.507856127415$$

70

The total exposure is $45+10+45+14+27+12+74+27+10+293+14+13+10+14+17+6=631$ units.

The means for each individual are: $\frac{34}{114}=0.2982456$, $\frac{0}{140}=0$, $\frac{169}{330}=0.5121212$, and $\frac{7}{47}=0.1489362$. The average value of λ (Using equal weighting of policyholders) is therefore $\frac{0.2982456+0+0.5121212+0.1489362}{4}=0.2398258$, so this is the expected process variance, because the variance of a Poisson distribution is equal to the mean.

Suppose the hypothetical means are λ_1, λ_2, λ_3, and λ_4, and denote our estimated means by $\widehat{\lambda_1}$, $\widehat{\lambda_2}$, $\widehat{\lambda_3}$, and $\widehat{\lambda_4}$. We have that $\mathbb{E}(\widehat{\lambda_i}|\lambda_i)=\lambda_i$ and $\mathrm{Var}(\widehat{\lambda_i}|\lambda_i)=\frac{\lambda_i}{n_i}$. Letting $\widehat{\lambda}=\frac{\widehat{\lambda_1}+\widehat{\lambda_2}+\widehat{\lambda_3}+\widehat{\lambda_4}}{4}$, we have that $\frac{(\widehat{\lambda_1}-\widehat{\lambda})^2+(\widehat{\lambda_2}-\widehat{\lambda})^2+(\widehat{\lambda_3}-\widehat{\lambda})^2+(\widehat{\lambda_4}-\widehat{\lambda})^2}{3}$ is an unbiased estimator for $\mathrm{Var}(\widehat{\lambda_i})$. Now we have that

$$\mathrm{Var}(\widehat{\lambda_i})=\mathrm{Var}(\mathbb{E}(\widehat{\lambda_i}|\lambda_i))+\mathbb{E}(\mathrm{Var}(\widehat{\lambda_i}|\lambda_i))=\mathrm{Var}(\lambda_i)+\mathbb{E}\left(\frac{\lambda_i}{n_i}\right)$$

We calculate

$$\mathrm{Var}(\widehat{\lambda_i})\frac{(0.2982456-0.2398258)^2+(0-0.2398258)^2+(0.5121212-0.2398258)^2+(0.1489362-0.2398258)^2}{3}$$
$$=0.04777833$$

We also calculate

$$\mathbb{E}\left(\frac{\lambda_i}{n_i}\right)=\frac{1}{4}\left(\frac{0.2398258}{114}+\frac{0.2398258}{140}+\frac{0.2398258}{330}+\frac{0.2398258}{47}\right)=0.00241154949013$$

The VHM is therefore $0.04777833-0.00241154949013=0.0453667805099$.

The credibility for the four policyholders is therefore given in the following table:

Policyholder	Exposure	Credibility
1	114	$\frac{114}{114+\frac{0.2398258}{0.0453667805099}}=0.955683331831$
2	140	$\frac{140}{140+\frac{0.2398258}{0.0453667805099}}=0.963614105621$
3	330	$\frac{330}{330+\frac{0.2398258}{0.0453667805099}}=0.984233255261$
4	37	$\frac{37}{37+\frac{0.2398258}{0.0453667805099}}=0.874986334869$

To balance the estimates, we set the book premium to equal the credibility-weighted mean

$$\frac{0.955683331831\times 0.2982456+0.963614105621\times 0+0.984233255261\times 0.5121212+0.874986334869\times 0.1489362}{0.955683331831+0.963614105621+0.984233255261+0.874986334869}=0.243320910699$$

The credibility estimate for λ_3 is therefore $0.984233255261\times 0.5121212+0.015766744739\times 0.243320910699=0.507883094453$.

71 For an exponential distribution, the variance is the square of the mean, so the estimate for EPV is the average square of the hypothetical means. This is the square of the mean claim amount plus the variance of hypothetical means. We have that the variance of observed means is 832^2. This is the VHM plus the EPV. Since EPV $=$ VHM $+ 689^2$, we have that $2\text{VHM} + 689^2 = 832^2$, so VHM $= \frac{832^2-689^2}{2} = 108751.5$ and EPV $= 108751.5 + 689^2 = 583472.5$. The credibility of one year's experience is therefore $Z = \frac{1}{1+\frac{583472.5}{108751.5}} = 0.157104492188$. The premium for this individual is therefore $0.157104492188 \times 462 + 0.842895507812 \times 689 =$ \$653.34.

20 Simulation

20.1 Basics of Simulation

20.2 Simulation for Specific Distributions

20.3 Determining the Sample Size

20.4 Examples of Simulation in Actuarial Modelling

IRLRPCI 3 Ratemaking

3.1 Introduction

3.2 Objectives of ratemaking

3.3 Frequency and severity

3.4 Data for ratemaking

3.5 Premium data

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(a) The premium period is from the 1st of October 2015 to the 30th of September 2016, so there are three months out of 12 in 2015. The earned premium is therefore $640 \times \frac{3}{12} = \$160$.

[If we divide by number of days, the earned premium is $640 \times \frac{91}{365} = \$159.56$.]

(b) There are nine months out of 12 in 2016, so the earned premium is $640 \times \frac{9}{12} = \$480$.

[If we divide by number of days, the earned premium is $640 \times \frac{274}{365} = \$480.44$.]

3.6 The exposure unit

3.7 The expected effective period

73

The new rate will cover annual policies written in 2018. These will be effective for 1 year. Assuming the time that policies are written is uniformly distributed over the year, the number of policies in force at the new rate t years from the start of 2018 is given by

$$\begin{cases} Nt & \text{if } t < 1 \\ N(2-t) & \text{if } 1 \leqslant t < 2 \\ 0 & \text{otherwise} \end{cases}$$

Assume that the rate of claims is proportional to the number of policies in force (so claims per policy does not depend on time of year). Assume also that claims in accident year 2016 were uniformly distributed over the year. Now the average claim time in accident year 2016 is the middle of the year. The average claim time in policy year 2018, is the end of 2018, which is 2.5 years later, so the inflation factor is $1.03^{2.5} = 1.076696$, so the average claim amount, accounting for inflation is $26000 \times 1.076696 = \$27,994.09$. The pure premium is therefore $27994.09 \times 0.003 = \$83.98$.

Alternatively: we could integrate over the inflation factor for all claim times. That is, if expected claim amount at the start of 2016 is C, then we have $C\int_0^1 (1.03)^t\,dt = 26000$, and we want to calculate

$$C(1.03)^2\left(\int_0^1 t(1.03)^t\,dt + \int_1^2 (2-t)(1.03)^t\,dt\right)$$

We have that

$$\int_0^1 (1.03)^t\,dt = \left[\frac{1.03^t}{\log(1.03)}\right]_0^1 = \frac{0.03}{\log(1.03)} = 1.014926$$

$$\int_0^1 t(1.03)^t\,dt = \left[t\frac{1.03^t}{\log(1.03)}\right]_0^1 - \int_0^1 \frac{1.03^t}{\log(1.03)}\,dt = \frac{1.03}{\log(1.03)} - \frac{0.03}{\log(1.03)^2} = 0.509963$$

$$\int_1^2 (2-t)(1.03)^t\,dt = 1.03\int_0^1 (1-s)(1.03)^s\,ds = 1.03\left(\frac{0.03}{\log(1.03)} - \frac{1.03}{\log(1.03)} + \frac{0.03}{\log(1.03)^2}\right) = 0.520112$$

$$\int_0^1 t(1.03)^t\,dt + \int_1^2 (2-t)(1.03)^t\,dt = \frac{0.03^2}{\log(1.03)^2}$$

We therefore get $1.014926C = 26000$ and the expected claim amount per claim for policy year 2018 is

$$(1.03)^2(0.509963 + 0.520112)C = \frac{(1.03)^2(0.509963 + 0.520112)}{1.014926} \times 26000 = 1.076735 \times 26000 = \$27,995.11$$

so the pure premium is $27995.11 \times 0.003 = \$83.99$.

[Algebraically, we can write the expected claim amount as $1.03^2 \times \frac{0.03}{\log(1.03)} \times 26000$.]

3.8 Ingredients of ratemaking

3.9 Rate Changes

74 For the base class, the loss ratio is $\frac{3900}{4100} = 0.9512195$. We want to change the current differentials to match this loss ratio. For example, for the low risk class, at a differential of 0.74, we get a loss ratio of $\frac{1100}{1300} = 0.8461538$. That is, if the premium were the same as for the base class, the loss ratio would be $\frac{1100}{\left(\frac{1300}{0.74}\right)} 0.8461538 \times 0.74 = 0.6261538$. To get this loss ratio to equal 0.9512195, we would need the new differential to be the solution to

$$\frac{\left(0.74 \times \frac{1100}{1300}\right)}{d} = \frac{3900}{4100}$$

or

$$d = 0.74 \times \frac{1100}{1300} \times \frac{4100}{3900} = 0.6582643$$

Similarly for the high risk class, the new differential is

$$1.46 \times \frac{1400}{1600} \times \frac{4100}{3900} = 1.343013$$

75

We first need to adjust the earned premiums to the new rate. Under uniform distribution, we have that $\frac{1}{2}\times\frac{3}{4}\times 34=\frac{9}{32}$ of the policies are at rate \$432; $\frac{1}{2}\times\frac{5}{12}\times 512=\frac{25}{288}$ of the policies are at rate \$491; and the remaining $\frac{182}{288}$ are at rate \$464. The average premium per policy is therefore $\frac{9}{32}\times 432+\frac{182}{288}\times 464+\frac{25}{288}\times 491=457.34375$. The adjusted earned premiums are therefore $1700000\times\frac{491}{457.34375}=1825104.20226$. The loss ratio at this premium is therefore $\frac{1520000}{1825104.20226}=0.832829160175$. The new premium before inflation is therefore $491\times\frac{0.832829160175}{0.8}=511.148897058$.

To calculate inflation from accident year 2018 to policy year 2020, we calculate $\int_0^1(1.04)^t dt=\left[\frac{(1.04)^t}{\log(1.04)}\right]_0^1=\frac{0.04}{\log(1.04)}=1.01986926764$ and

$$\begin{aligned}\int_0^1 t(1.04)^t\,dt+\int_1^2(2-t)(1.04)^t\,dt &= \int_0^1 t(1.04)^t\,dt+1.04\int_0^1(1-t)(1.04)^t\,dt\\ &= 1.04\int_0^1(1.04)^t\,dt-0.04\int_0^1 t(1.04)^t\,dt\\ &= 1.04\times 1.01986926764-0.04\left(\left[\frac{t(1.04)^t}{\log(1.04)}\right]_0^1-\int_0^1\frac{(1.04)^t}{\log(1.04)}\,dt\right)\\ &= 1.06066403835-\frac{0.04\times 1.04}{\log(1.04)}+\frac{0.04\times 1.01986926764}{\log(1.04)}\\ &= 1.04013332308\end{aligned}$$

The inflation is therefore $\frac{1.04^2\times 1.04013332308}{1.01986926764}=1.10309059988$, so the new premium is $1.10309059988\times 511.148897058=\$563.84$

76 Loss Ratio Method:

The new differentials are Low risk $0.74\times\frac{1900}{2000}\times\frac{9100}{8000}=0.7996625$ High risk $1.46\times\frac{2300}{3300}\times 91008000=1.157492$ Female : $0.88\times\frac{5850}{6900}\times\frac{7500}{6350}=0.8812051$.

The permissible loss ratio is $1-0.2=80\%$. At the current premium, the loss ratio is $\frac{12200}{14400}=0.8472222$, so if they used the same relative changes to premiums, the premiums would change by a factor of $\frac{0.8472222}{0.8}=1.059028$. However, we want to balance back by dividing by the off-balance factor which is

$$\frac{\frac{0.7997}{0.74}\times 900+1\times 4700+\frac{1.1575}{1.46}\times 1900+\frac{0.7997}{0.74}\times\frac{0.8812}{0.88}\times 1100+\frac{0.8812}{0.88}\times 4400+\frac{1.1575}{1.46}\times\frac{0.8812}{0.88}\times 1400}{900+4700+1900+1100+4400+1400}=0.9643522$$

We therefore multiply the base rate by $\frac{1.059028}{0.9643522}=1.123013$. The new base rate is $1.098176\times 46.30=\$50.85$. The rates for other classes are therefore shown in the following table

	Male	Female
Low	40.66	35.83
Medium	50.85	44.81
High	58.85	51.86

We now compare the calculated differentials with the experience:

Calculated Differential				Experience	
		Male	Female	Male	Female
Differential		1	0.8812051		
Low	0.7996625	0.7996625	0.7046667	0.9896748	0.5768390
Medium	1	1.0000000	0.8812051	1.0000000	0.8941463
High	1.157492	1.1574920	1.0199879	1.0570475	1.1572153

Loss cost method:

To calculate the new differentials, we calculate the loss cost per unit of exposure for each class. For example for female policyholders, the total loss was \$5,850, and there were \$1,100 of earned premiums at a rate of $46.30\times 0.74\times 0.88$, which corresponds to $\frac{1100}{46.30\times 0.74\times 0.88}=36.48357$ units of exposure. Similarly, the number of units of exposure for the other classes were:

	Male	Female	total
Low	26.2682	36.4836	62.7517
Medium	101.5119	107.9914	209.5032
High	28.1073	23.5349	51.6422
total	155.8874	168.0098	

This gives the loss costs as

Class	Loss cost
Low	30.27804
Medium	38.18557
High	44.53722
Male	40.73453
Female	34.81940

This gives the following:

Calculated Differential				Experience	
		Male	Female	Male	Female
Differential		1	0.8547883		
Low	0.7929184	0.7929184	0.6777774	0.9896748	0.5768390
Medium	1	1.0000000	0.8547883	1.0000000	0.8941463
High	1.166336	1.1663360	0.9969704	1.0570475	1.1572153

For these differentials and the exposures calculated above, the total earned premiums would be 295.624 times the base rate. The expected total losses are \$12,200, so the new base rate is $\frac{12200}{0.8\times 295.624} = \$51.59$, and the new premiums are

	Male	Female
Low	40.90	34.96
Medium	51.59	44.09
High	60.17	51.43

IRLRPCI 5 Intermediate topics

5.1 Individual risk rating plans

77

For the \$50,000 to \$100,000 increase, we consider only the policies with limit at least \$100,000. For these policies, the total losses limited to \$100,000 are $41,000 + 26000 + 12300 = 79300$, while total losses limited to \$50,000 for the same policies are $34000 + 23000 + 11000 = 68000$. The ILF is therefore $\frac{79300}{68000} = 1.166176$

For the \$100,000 to \$500,000 increase, the ILF is $\frac{31000+13400}{26000+12300} = \frac{44400}{38300} = 1.159269$. The other ILFs are given in the following table

Old Policy limit	New Policy Limit		
	100,000	500,000	1,000,000
50,000	1.166176	1.305882	1.545455
100,000		1.159269	1.382114
500,000			1.268657

If instead, we use incremental factors, the incremental factors are on the diagonal of the above table, and other factors are obtained by multiplying the incremental factors below them, so for example, the ILF from \$50,000 to \$500,000 would be $1.166176 \times 1.159269 = 1.351912$, while the ILF for an increase from \$50,000 to \$1,000,000 is $1.166176 \times 1.159269 \times 1.268657 = 1.715112$. The ILFs based on this incremental method are given in the following table:

Old Policy limit	New Policy Limit		
	100,000	500,000	1,000,000
50,000	1.166176	1.351912	1.715112
100,000		1.159269	1.470715
500,000			1.268657

78

The expected payments per claim are $a\theta - \mathbb{E}((X-5a\theta)_+)$. By the memoryless property of the exponential distribution $\mathbb{E}((X-5a\theta)_+) = a\theta P(X > 5a\theta) = a\theta e^{-\frac{5a\theta}{a\theta}} = a\theta e^{-5}$. The expected payment per claim is therefore $(1-e^{-5})a\theta$.

After inflation, the new mean loss per claim is $1.1a\theta$. The expected payment per claim is $1.1a\theta - \mathbb{E}((X - 5a\theta)_+)$, and $\mathbb{E}((X - 5a\theta)_+) = 1.1a\theta e^{-\frac{5a\theta}{1.1a\theta}} = 1.1e^{-\frac{5}{1.1}}a\theta$. The new expected claim payment is therefore $1.1a\theta(1-e^{-\frac{5}{1.1}})$. The percentage increase in claim payments is therefore $\frac{1.1(1-e^{-\frac{5}{1.1}})}{1-e^{-5}} - 1 = 9.5706\%$.

79 We will use a normal approximation to aggregate losses.
(a) The expected value of a Pareto distribution censored at \$50,000 is

$$10000\int_0^5 \frac{1}{(1+u)^3}\,du = 10000\int_1^6 a^{-3}\,da = 10000\left[-\frac{a^{-2}}{2}\right]_1^6 = \frac{10000}{2}\left(1-\frac{1}{6^2}\right) = \$4,861.11$$

The expected square of the payment censored at \$50,000 is

$$10000^2\int_0^5 \frac{2u}{(1+u)^3}\,du = 10000^2\int_1^6 2(a-1)a^{-3}\,da = 10000^2\left[a^{-2}-2a^{-1}\right]_1^6 = 10000^2\left(1+\frac{1}{6^2}-\frac{2}{6}\right) = 69,444,444$$

so the variance is $69444444 - 4861.11^2 = 45,814,043$.

The mean aggregate loss is therefore $100 \times 4861.11 = \$486,111.11$ and the variance of aggregate loss is $100 \times 45814043 + 100 \times 4861.11^2 = 6,944,444,444$. Using a normal approximation, the 95th percentile of aggregate losses is $486,111.11 + 1.645\sqrt{6,944,444,444} = \$623182.25$. The risk loading as a percentage of the gross rate is therefore $\frac{623182.25-486111.11}{623182.25} = 21.99535\%$.

(b) Censored at \$100,000, the expected payment per loss is

$$10000\int_0^{10} \frac{1}{(1+u)^3}\,du = 10000\int_1^{11} a^{-3}\,da = 10000\left[-\frac{a^{-2}}{2}\right]_1^{11} = \frac{10000}{2}\left(1-\frac{1}{11^2}\right) = \$4,958.69$$

The expected squared loss is

$$10000^2\int_0^{10} \frac{2u}{(1+u)^3}\,du = 10000^2\int_1^{11} 2(a-1)a^{-3}\,da = 10000^2\left[a^{-2}-2a^{-1}\right]_1^{11} = 10000^2\left(1+\frac{1}{11^2}-\frac{2}{11}\right) = 82,644,628$$

and the variance is $82644628 - 4958.69^2 = 58056144$.

The mean aggregate loss is $100 \times 4958.68 = \$495,867.77$ and the variance is $100 \times 82644628 = 8264462810$. The 95th percentile is then $495,867.77 + 1.645\sqrt{8264462810} = \$645399.92$. The risk loading as a percentage of the gross rate is therefore $\frac{645399.92-495867.77}{645399.92} = 23.16891\%$.

80

The total number of claims is 3365. We calculate the average loss for each policy limit. For example for policy limit \$10,000, the total claims would be $6850000 + 1065 \times 10000 = \$17,500,000$, so the average claim would be $\frac{17500000}{3365} = \$5,200.59$.

Policy limit	Total claimed	Average per claim	Risk Charge	Total charge
100,000	36,950,000	10980.68	2411.51	\$13,392.19
500,000	52,350,000	15557.21	4840.53	\$20,397.74
1,000,000	58,450,000	17369.99	6034.33	\$23,404.31

The ILF from \$100,000 to \$500,000 is $\frac{20397.74}{13392.19} = 1.523107$. and to \$1,000,000, it is $\frac{23404.31}{13392.19} = 1.747609$.

[For the pure premium, the ILFs would be 1.416779 and 1.581867 respectively.]

81

If the attachment point is a, then the expected aggregate loss is $\theta(1-e^{-\frac{a}{\theta}})$, and the expected claim on the stop-loss insurance is $\theta e^{-\frac{a}{\theta}}$. The variance of the aggregate loss payment is

$$\begin{aligned}
\theta^2\left(\int_0^{\frac{a}{\theta}} x^2e^{-x}\,dx+\frac{a^2}{\theta^2}e^{-\frac{a}{\theta}}-(1-e^{-\frac{a}{\theta}})^2\right) &= \theta^2\left(\left[-x^2e^{-x}\right]_0^{\frac{a}{\theta}}+2\int_0^{\frac{a}{\theta}} xe^{-x}\,dx+\frac{a^2}{\theta^2}e^{-\frac{a}{\theta}}-1+2e^{-\frac{a}{\theta}}-e^{-2\frac{a}{\theta}}\right)\\
&= \theta^2\left(-\frac{a^2}{\theta^2}e^{-\frac{a}{\theta}}+2\left[-xe^{-x}\right]_0^{\frac{a}{\theta}}+2\int_0^{\frac{a}{\theta}} e^{-x}\,dx+\frac{a^2}{\theta^2}e^{-\frac{a}{\theta}}-1+2e^{-\frac{a}{\theta}}-e^{-2\frac{a}{\theta}}\right)\\
&= \theta^2\left(-\frac{a^2}{\theta^2}e^{-\frac{a}{\theta}}-2\frac{a}{\theta}e^{-\frac{a}{\theta}}+2\left(1-e^{-\frac{a}{\theta}}\right)+\frac{a^2}{\theta^2}e^{-\frac{a}{\theta}}-1+2e^{-\frac{a}{\theta}}-e^{-2\frac{a}{\theta}}\right)\\
&= \theta^2\left(1-2\frac{a}{\theta}e^{-\frac{a}{\theta}}-e^{-2\frac{a}{\theta}}\right)
\end{aligned}$$

The insurer's premium is therefore set at

$$P=\theta\left(2e^{-\frac{a}{\theta}}+1-e^{-\frac{a}{\theta}}+\sqrt{1-2\frac{a}{\theta}e^{-\frac{a}{\theta}}-e^{-2\frac{a}{\theta}}}\right)$$

We want to minimise this P. We substitute $u=\frac{a}{\theta}$ and calculate

$$\frac{dP}{du}=\theta\left(-e^{-u}+\frac{ue^{-u}-e^{-u}+e^{-2u}}{\sqrt{1-2ue^{-u}-e^{-2u}}}\right)$$

We find the minimum by setting this equal to zero:

$$\begin{aligned}
-e^{-u}+\frac{ue^{-u}-e^{-u}+e^{-2u}}{\sqrt{1-2ue^{-u}-e^{-2u}}} &= 0\\
\frac{ue^{-u}-e^{-u}+e^{-2u}}{\sqrt{1-2ue^{-u}-e^{-2u}}} &= e^{-u}\\
\frac{(ue^{-u}-e^{-u}+e^{-2u})^2}{1-2ue^{-u}-e^{-2u}} &= e^{-2u}\\
(u-1+e^{-u})^2 &= 1-2ue^{-u}-e^{-2u}\\
u^2-2u+4ue^{-u}-2e^{-u}+2e^{-2u} &= 0
\end{aligned}$$

Numerically, we solve this to get $u = 0.9640863$, so the attachment point that minimises the premium is 0.9640863 times the mean aggregate claims.