ACSC/STAT 4703, Actuarial Models II

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Toby Kenney

Homework Sheet 1

Model Solutions

Basic Questions

1. *Aggregate payments have a compund distribution. The frequency distribution is Binomial with $n = 7$ and $p = 0.4$. The severity distribution is Pareto with shape $\alpha = 2.3$ and scale $\theta = 600$. Use a Pareto approximation to aggregate payments to estimate the probability that aggregate payments are more than 10,000.*

   The frequency distribution has mean $7 \times 0.4 = 2.8$ and variance $7 \times 0.4 \times 0.6 = 1.68$. The severity distribution has mean $\frac{600}{1.3} = 461.538461538$ and variance $\frac{600^2 \times 2.3}{1.3^2 \times 0.3} = 1633136.09467$

   The mean of aggregate losses is given by $2.8 \times 461.538461538 = 1292.30769231$ and variance $2.8 \times 1633136.09467 + 1.68 \times 461.538461538^2 = 4930650.88757$. Setting these equal to the mean and variance of a Pareto distribution with parameters α and θ gives

$$\frac{\theta}{\alpha - 1} = 1292.30769231$$
$$\frac{\alpha\theta^2}{(\alpha-1)^2(\alpha-2)} = 4930650.88757$$
$$\frac{\alpha}{\alpha - 2} = \frac{4930650.88757}{1292.30769231^2} = 2.95238095237$$
$$\alpha = 3.02439024391$$
$$\theta = 2616.13508444$$

   For these parameters, the probability that payments exceed \$10,000 is $\left(\frac{2616.13508444}{2616.13508444+10000}\right)^{3.02439024391} = 0.0085809718139$.

2. *Loss amounts follow a Gamma distribution with shape $\alpha = 4.6$ and scale $\theta = 500$. The distribution of the number of losses is given in the following table:*

Number of Losses	*Probability*
0	*0.880*
1	*0.064*
2	*0.035*
3	*0.021*

Assume all losses are independent and independent of the number of losses. The insurance company buys excess-of-loss reinsurance on the part of the loss above $1,000. Calculate the expected payment for this excess-of-loss reinsurance.

If there are n claims, then the total losses follow a gamma distribution with shape $\alpha = 4.6n$ and $\theta = 500$. The expected payment on the excess of loss distribution in this case is therefore given by:

$$\begin{aligned}
\mathbb{E}((X-1000)_+) &= \frac{500}{\Gamma(4.6n)}\int_2^\infty (x-2)x^{4.6n-1}e^{-x}\,dx \\
&= \frac{500}{\Gamma(4.6n)}\left(\int_2^\infty x^{4.6n}e^{-x}\,dx - 2\int_2^\infty x^{4.6n-1}e^{-x}\,dx\right) \\
&= 500\left(\frac{4.6n}{\Gamma(4.6n+1)}\int_2^\infty x^{4.6n}e^{-x}\,dx - \frac{2}{\Gamma(4.6n)}\int_2^\infty x^{4.6n-1}e^{-x}\,dx\right) \\
&= 500\left(4.6n\texttt{pgamma(2,shape=4.6n+1,lower.tail=FALSE)} - 2\texttt{pgamma(2,shape=4.6n+1,lower.tail=FALSE)}\right)
\end{aligned}$$

This gives the following table

n	$P(N=n)$	$\mathbb{E}((S-1000)_+\vert N=n)$	$\mathbb{E}((S-1000)_+ I_{N=n})$
0	0.880	0.000	0.000
1	0.064	1318.512	84.385
2	0.035	3600.020	126.001
3	0.021	5900.000	123.900

So the total expected payment is $84.385 + 126.001 + 123.900 = \$334.29$

3. *Claim frequency follows a negative binomial distribution with* $r = 1.8$ *and* $\beta = 2.2$. *Claim severity (in thousands) has the following distribution:*

Severity	*Probability*
1	*0.69*
2	*0.24*
3	*0.06*
4 or more	*0.01*

Use the recursive method to calculate the exact probability that aggregate claims are at least $4,000.

Since the severity distribution is zero-truncated, the aggregate loss distribution is zero only if claim frequency is zero, which has probability

$\frac{1}{(1+2.2)^{1.8}} = 0.123233856344$. Recall that for the negative binomial distribution, $a = \frac{\beta}{1+\beta} = \frac{2.2}{3.2}$ and $b = \frac{(r-1)\beta}{1+\beta} = \frac{1.76}{3.2}$.

The recurrence formula is

$$f(x) = \sum_{k=1}^{x} \frac{2.2}{3.2}\left(1 + \frac{0.8k}{x}\right) f_X(k) f(x-k)$$

Applying this gives:

$$f(1) = \frac{2.2}{3.2} \times 1.8 \times 0.69 \times 0.123233856344 = 0.105226309086$$

$$f(2) = \frac{2.2}{3.2}(1.4 \times 0.69 \times 0.105226309086 + 1.8 \times 0.24 \times 0.123233856344) = 0.106483877856$$

$$f(3) = \frac{2.2}{3 \times 3.2}(3.8 \times 0.69 \times 0.106483877856 + 4.6 \times 0.24 \times 0.105226309086 + 5.4 \times 0.06 \times 0.12323385634$$

The probability that aggregate claims are at least \$4,000 is therefore

$$\begin{aligned} & 1 - f(0) - f(1) - f(2) - f(3) \\ =& 1 - 0.123233856344 - 0.105226309086 - 0.106483877856 - 0.0997558701392 \\ =& 0.565300086575 \end{aligned}$$

4. *Use an arithmetic distribution ($h = 1$) to approximate a Weibull distribution with shape $\tau = 0.5$ and scale $\theta = 8.2$.*

   (a) Using the method of rounding, calculate the mean of the arithmetic approximation. [You can evaluate this numerically: use 2,000 terms in the sum.]

   Using the method of rounding, we set $p_0 = P\left(X < \frac{1}{2}\right)$ and $p_n = P\left(n - \frac{1}{2} < X < n + \frac{1}{2}\right)$ for $n > 0$. For the Weibull distribution, we have $P(X > x) = e^{-\left(\frac{x}{8.2}\right)^{0.5}}$, so

$$\begin{aligned} \mathbb{E}(X_a) &= \sum_{n=1}^{\infty} P(X_a \geqslant n) \\ &= \sum_{n=1}^{\infty} P\left(X > n - \frac{1}{2}\right) \\ &= \sum_{n=1}^{\infty} e^{-\left(\frac{2n-1}{16.4}\right)^{0.5}} \end{aligned}$$

   We compute this in `R`.

   This gives $\mathbb{E}(X_a) = 16.38116$.

   (b) Using the method of local moment matching, matching 1 moment on each interval, estimate the probability that the value is larger than 5.5.

We have

$$p_0+p_1+p_2+p_3+p_4+p_{5,l}=1-e^{-\left(\frac{5}{8.2}\right)^{0.5}}=0.5419921$$

and

$$\begin{aligned}
p_{5,u}+p_{6,l}&=e^{-\left(\frac{5}{8.2}\right)^{0.2}}-e^{-\left(\frac{6}{8.2}\right)^{0.2}}\\
&=0.03289435\\
5p_{5,u}+6p_{6,l}&=\int_5^6 x\times 0.5\left(\frac{x^{-0.5}}{8.2^{0.5}}\right)e^{-\left(\frac{x}{8.2}\right)^{0.5}}\,dx\\
&=0.5\times 8.2^{-0.5}\int_5^6 ue^{-\frac{u}{8.2^{0.5}}}\,dx\\
&=8.2^{-0.5}\int_{5^{0.5}}^{6^{0.5}} u^2e^{-\frac{u}{8.2^{0.5}}}\,du\\
&=8.2^{-0.5}\left(\left[-8.2^{0.5}u^2e^{-\frac{u}{8.2^{0.5}}}\right]_{5^{0.5}}^{6^{0.5}}+2\times 8.2^{0.5}\int_{5^{0.5}}^{6^{0.5}} ue^{-\frac{u}{8.2^{0.2}}}\,du\right)\\
&\dots\\
&=8.2^{-0.5}\left(8.2^{0.5}\left(5e^{-\frac{5^{0.5}}{8.2^{0.5}}}-6e^{-\frac{6^{0.5}}{8.2^{0.5}}}\right)+2\times 8.2^{0.5}\left(5^{0.5}e^{-\frac{5^{0.5}}{8.2^{0.5}}}-6^{0.5}e^{-\frac{6^{0.5}}{8.2^{0.5}}}\right)+\dots\right)\\
&=e^{-\frac{5^{0.5}}{8.2^{0.5}}}\left(5+2\times 8.2^{0.5}\times 5^{0.5}+2\times 1\times 8.2\right)-e^{-\frac{6^{0.5}}{8.2^{0.5}}}\left(6+2\times 8.2^{0.5}\times 6^{0.5}+2\times 1\times 8.2\right)\\
&=0.1804649
\end{aligned}$$

So

$$p_{5,u}=6\times 0.03289435-0.1804649=0.0169012$$

Thus, $P(X_a>5.5)=1-0.5419921-0.0169012=0.4411067$

Standard Questions

5. *An insurance company models loss frequency as Poisson with parameter $\lambda=6$, and loss severity as Pareto with shape $\alpha=2.5$ and scale $\theta=2400$. One reinsurance company uses a gamma distribution to model aggregate losses, fitted by the method of moments, and sells stop-loss reinsurance with attachment point $10,000 for a loading of 100% based on the estimated payments under this model. Another reinsurance company uses a Pareto distribution to model aggregate losses and charges a 20% loading.*

   (a) Which reinsurance company is cheaper if each policy includes a deductible of $500.

With a deductible of \$500, the expected payment for each loss is

$$\begin{aligned}
\int_{500}^{\infty}\left(\frac{2400}{2400+x}\right)^{2.5}dx &= \int_{2900}^{\infty}2400^3u^{-2.5}\,du\\
&= \frac{2400^{2.5}}{1.5}\left[-u^{-1.5}\right]_{2900}^{\infty}\\
&= \frac{2400^{2.5}}{1.5\times 2900^{1.5}}\\
&= 1204.59164994
\end{aligned}$$

and the expected squared payment for each loss is

$$\begin{aligned}
\int_{500}^{\infty}2(x-500)\left(\frac{2400}{2400+x}\right)^{2.5}dx &= \int_{2900}^{\infty}2(u-2900)2400^{2.5}u^{-2.5}\,du\\
&= 2\times 2400^{2.5}\int_{2900}^{\infty}(u^{-1.5}-2900u^{-2.5})\,du\\
&= 2\times 2400^{2.5}\left(\frac{1}{0.5\times 2900^{0.5}}-\frac{2900}{1.5\times 2900^{1.5}}\right)\\
&= \frac{2\times 2400^{2.5}}{\frac{3}{8}\times 2900^{0.5}}\\
&= 27946526.2786
\end{aligned}$$

For the aggregate loss distribution, the expected aggregate payment is $6\times 1204.59164994 = 7227.54989964$ and the expected squared aggregate loss is $6\times 27946526.2786 = 167679157.672$. The variance of aggregate loss is $167679157.672 - 7227.54989964^2 = 115441680.12$.

For the gamma distribution, the estimated parameters are given by solving

$$\begin{aligned}
\alpha\theta &= 7227.54989964\\
\alpha\theta^2 &= 115441680.12\\
\theta &= \frac{115441680.12}{7227.54989964} = 15972.4501004\\
\alpha &= \frac{7227.54989964}{15972.4501004} = 0.452501016075
\end{aligned}$$

The expected payment on the reinsurance with attachment point $a = 10000$ is

$$
\begin{aligned}
&\int_a^\infty (x-a)\frac{x^{\alpha-1}e^{-\frac{x}{\theta}}}{\theta^\alpha\Gamma(\alpha)}\,dx\\
=&\int_a^\infty \frac{x^{\alpha}e^{-\frac{x}{\theta}}}{\theta^\alpha\Gamma(\alpha)}\,dx - a\int_a^\infty \frac{x^{\alpha-1}e^{-\frac{x}{\theta}}}{\theta^\alpha\Gamma(\alpha)}\,dx\\
=&\alpha\theta\int_a^\infty \frac{x^{\alpha}e^{-\frac{x}{\theta}}}{\theta^{\alpha+1}\Gamma(\alpha+1)}\,dx - a\int_a^\infty \frac{x^{\alpha-1}e^{-\frac{x}{\theta}}}{\theta^\alpha\Gamma(\alpha)}\,dx\\
=&\ 4239.77
\end{aligned}
$$

Thus, the premium is $2\times 4239.77 = \$8,479.54$.

For the company using the Pareto approximation, the estimated parameters are given by solving

$$
\begin{aligned}
\frac{\theta}{\alpha-1} &= 7227.54989964\\
\frac{\alpha\theta^2}{(\alpha-1)^2(\alpha-2)} &= 115441680.12\\
\frac{\alpha}{\alpha-2} &= \frac{115441680.12}{7227.54989964^2} = 2.20993978902\\
\alpha &= \frac{2}{1-\frac{1}{2.20993978902}} = 3.6529748159\\
\theta &= 7227.54989964\times 4.6529748159 = 33629.6076637
\end{aligned}
$$

Using this approximation, the expected payment is

$$
\begin{aligned}
\int_a^\infty \left(\frac{\theta}{\theta+x}\right)^\alpha dx &= \int_{\theta+a}^\infty \theta^\alpha u^{-\alpha}\,du\\
&= \theta^\alpha\left[\frac{u^{1-\alpha}}{1-\alpha}\right]_{a+\theta}^\infty\\
&= \frac{\theta^\alpha}{(\alpha-1)(a+\theta)^{\alpha-1}}\\
&= 6353.963
\end{aligned}
$$

Thus, the premium is $1.2\times 6353.963 = \$7,624.76$.

Thus, the second reinsurer is cheaper.

(b) Show that a deductible $d = 519.285$ *is a local maximum for the second reinsurer's premium.*

If the deductible is d, then the expected payment for each loss is

$$\begin{aligned}\int_d^\infty \left(\frac{2400}{2400+x}\right)^{2.5}\,dx &= \int_{2400+d}^\infty 2400^3 u^{-2.5}\,du\\ &= \frac{2400^{2.5}}{1.5}\left[-u^{-1.5}\right]_{2400+d}^\infty\\ &= \frac{2400^{2.5}}{1.5\times(2400+d)^{1.5}}\end{aligned}$$

and the expected squared payment for each loss is

$$\begin{aligned}\int_d^\infty 2(x-d)\left(\frac{2400}{2400+x}\right)^{2.5}\,dx &= \int_{2400+d}^\infty 2(u-(2400+d))2400^{2.5}u^{-2.5}\,du\\ &= 2\times 2400^{2.5}\int_{2400+d}^\infty (u^{-1.5}-(2400+d)u^{-2.5})\,du\\ &= 2\times 2400^{2.5}\left(\frac{1}{0.5\times(2400+d)^{0.5}}-\frac{2400+d}{1.5\times(2400+d)^{1.5}}\right)\\ &= \frac{2\times 2400^{2.5}}{\frac{3}{8}\times(2400+d)^{0.5}}\end{aligned}$$

The expected aggregate loss and expected squared aggregate loss are therefore

$$\frac{\theta}{\alpha-1} = 6\times\frac{2400^{2.5}}{1.5\times(2400+d)^{1.5}} = 4\times\frac{2400^{2.5}}{(2400+d)^{1.5}}$$

and

$$\frac{\alpha\theta^2}{(\alpha-1)^2(\alpha-2)} = 6\times\frac{2\times 2400^{2.5}}{\frac{3}{8}\times(2400+d)^{0.5}} = 32\times\frac{2400^{2.5}}{(2400+d)^{0.5}}$$

Thus, for the Pareto approximation, we have

$$\frac{\alpha}{\alpha-2} = \frac{32\times\frac{2400^{2.5}}{(2400+d)^{0.5}}}{\left(4\times\frac{2400^{2.5}}{(2400+d)^{1.5}}\right)^2} = \frac{2\times(2400+d)^{2.5}}{2400^{2.5}}$$

which gives $\alpha = \frac{2}{1-\frac{2400^{2.5}}{2\times(2400+d)^{2.5}}} = \frac{4\times(2400+d)^{2.5}}{2\times(2400+d)^{2.5}-2400^{2.5}} = 2+\frac{2\times 2400^{2.5}}{2\times(2400+d)^{2.5}-2400^{2.5}} =$ 2.88355945774
$\alpha = 2+\frac{2\times 2400^{2.5}}{2\times(2400+d)^{2.5}-2400^{2.5}}$ and

$$\theta = \frac{2\times(2400+d)^{2.5}+2400^{2.5}}{2\times(2400+d)^{2.5}-2400^{2.5}}\times 4\times\frac{2400^{2.5}}{(2400+d)^{1.5}} = 13478.8452104$$

From Part (a), the premium is

$$P = \frac{1.2\theta^\alpha}{(\alpha - 1)(a + \theta)^{\alpha - 1}}$$

$$\log(P) = \log(1.2) + \alpha \log(\theta) - \log(\alpha - 1) - (\alpha - 1) \log(a + \theta)$$

$$\frac{2}{\alpha} = 1 - \frac{2400^{2.5}}{2 \times (2400 + d)^{2.5}}$$

$$\frac{2}{\alpha^2} \frac{\partial \alpha}{\partial d} = -\frac{1.25 \times 2400^{2.5}}{(2400 + d)^{3.5}}$$

$$\frac{\theta}{\alpha - 1} = 4 \times \frac{2400^{2.5}}{(2400 + d)^{1.5}}$$

$$\frac{1}{\alpha - 1} \frac{d\theta}{dd} - \frac{\theta}{(\alpha - 1)^2} \frac{d\alpha}{dd} = -6 \times \frac{2400^{2.5}}{(2400 + d)^{2.5}}$$

We calculate

$\alpha = 2.88355945774$

$\theta = 13478.8452104$

$$
\begin{aligned}
\frac{\partial \log(P)}{\partial \alpha} &= \log(\theta) - \log(a+\theta) - \frac{1}{\alpha-1} \\
&= -0.926027490552 \\
\frac{\partial \log(P)}{\partial \theta} &= \frac{\alpha}{\theta} - \frac{\alpha-1}{a+\theta} \\
&= 0.0000834807177338 \\
\frac{d\alpha}{dd} &= -\alpha^2 \frac{5 \times 2400^{2.5}}{4(2400+d)^{3.5}} \\
&= -80\alpha^2(\alpha-1)(\alpha-2)^2\theta^{-1} \\
&= -0.0821315911608 \\
\frac{1}{\alpha-1}\frac{d\theta}{dd} + 80\frac{\theta}{(\alpha-1)^2}\alpha^2(\alpha-1)(\alpha-2)^2\theta^{-1} &= -6 \times \frac{2400^{2.5}}{(2400+d)^{2.5}} \\
&= -12\frac{\alpha-2}{\alpha} \\
\frac{d\theta}{dd} &= -80\alpha^2(\alpha-2)^2 - 12\frac{(\alpha-1)(\alpha-2)}{\alpha} \\
&= -187.016116675 \\
\frac{d\log(P)}{dd} &= \frac{\partial \log(P)}{\partial \alpha}\frac{d\alpha}{dd} + \frac{\partial \log(P)}{\partial \theta}\frac{d\theta}{dd} \\
&= -0.926027490552 \times -0.0821315911608 + 0.000083480717 \\
&= 0.0604438716099
\end{aligned}
$$

Substituting these results, we get

$$
\begin{aligned}
\frac{d\log(P)}{dd} &= \frac{\partial \log(P)}{\partial \alpha}\frac{d\alpha}{dd} + \frac{\partial \log(P)}{\partial \theta}\frac{d\theta}{dd} \\
&= \left(\log\left(\frac{\theta}{a+\theta}\right) - \frac{1}{\alpha-1}\right)\left(-80\alpha^2(\alpha-1)(\alpha-2)^2\theta^{-1}\right) \\
&\qquad + \left(\frac{\alpha}{\theta} - \frac{\alpha-1}{a+\theta}\right)\left(-80\alpha^2(\alpha-2)^2 - 12\frac{(\alpha-1)(\alpha-2)}{\alpha}\right) \\
&= -80\frac{\alpha^2(\alpha-1)(\alpha-2)^2}{\theta}\log\left(\frac{\theta}{a+\theta}\right) + \frac{80\alpha^2(\alpha-2)^2}{\theta}(1-\alpha) - \frac{12\alpha(\alpha-2)}{\theta} \\
&\qquad + \frac{\alpha-1}{a+\theta}\left(-80\alpha^2(\alpha-2)^2 - 12\frac{(\alpha-1)(\alpha-2)}{\alpha}\right)
\end{aligned}
$$

6. *The number of claims an insurance company receives follows a binomial distribution with* $n = 128$ *and* $p = 0.64$*. Claim severity follows a negative binomial distribution with* $r = 8.3$ *and* $\beta = 24$*. Calculate the probability that aggregate losses exceed \$20,000.*

(a) Starting the recurrence 6 standard deviations below the mean [You need to calculate 30000 terms of the recurrence for f_s.]

Claim frequency has mean $128 \times 0.64 = 81.92$ and variance $128 \times 0.64 \times 0.36 = 29.4912$. Claim severity has mean $8.3 \times 24 = 199.2$ and variance $8.3 \times 24 \times 25 = 4980$. Aggregate losses therefore have mean $81.92 \times 199.2 = 16318.464$ variance $81.92 \times 4980 + 29.4912 \times 199.2^2 = 1578191.29037$. This means that 6 standard deviations below the mean is $16318.464 - 6\sqrt{1578191.29037} = 8780.89897848$ We therefore start the recurrence at $x = 8780$.

For the binomial distribution with $n = 128$ and $p = 0.64$, we have $a = -\frac{p}{1-p} = -\frac{0.64}{0.36} = -\frac{16}{9}$ and $b = (n+1)\frac{p}{1-p} = 129 \times \frac{16}{9} = \frac{688}{3}$. The recurrence is therefore

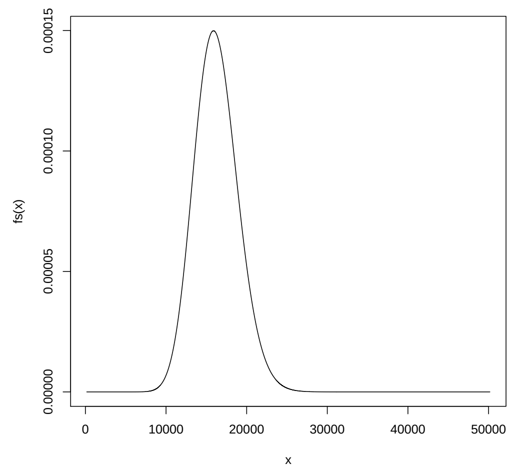
$$f_S(x) = \frac{1}{1 + \frac{16}{9} f_X(0)} \sum_{y=1}^{x} \frac{16}{9}\left(\frac{129y}{x} - 1\right) f_X(y) f_S(x-y)$$

$$= \frac{1}{1 + \frac{16}{9} \times 25^{-8.3}} \sum_{y=1}^{x} \frac{16}{9}\left(\frac{129y}{x} - 1\right) 25^{-8.3} \binom{y+7.3}{y} \left(\frac{24}{25}\right)^y f_S(x-y)$$

```
n<-seq_len(30000)
fx<-choose(n+7.3,n)*(24/25)^n/25^(8.3)
#define a vector of the secondary distribution.

fs<-n #prepare a vector to store results
for(i in n){
    y<-seq_len(i)
    x<-i+8780
    fs[i+1]<-sum((129*y/x-1)*fx[y]*fs[i+1-y])*16/(9+16*25^(-8.3))
}
fs<-fs/sum(fs)

# Now fs[i]=fs(8779+i)
sum(fs[(20001-8779):30000])
#question asks for strict inequality.
```



This gives the probability that $S > 20000$ as 0.001810746

(b) Using a suitable convolution.

If $N \sim B(128, 0.64)$, we can say $N = N_1 + N_2 + \cdots + N_8$ with $N_i \sim B(16, 0.64)$. This gives $S = S_1 + \cdots + S_8$, where $S_1 = X_1 + \cdots + X_{N_1}$ and so on. We therefore compute the distribution of each S_i using the recurrence:

$$f_{S_i}(x) = \frac{1}{1 + \frac{16}{9} \times 25^{-8.3}} \sum_{y=1}^{x} \frac{16}{9} \left(\frac{17y}{x} - 1 \right) 25^{-8.3} \binom{y+7.3}{y} \left(\frac{24}{25} \right)^y f_S(x-y)$$

We calculate $f_{S_i}(0) = P_{S_i}(f_X(0)) = \left(0.36 + 0.64 \times 25^{-8.3}\right)^{16} = 7.95866111065 \times 10^{-8}$.

```
g<-rep(0,15001)
g[1]=(0.36+0.64/25^(8.3))^(16) #f_{S_i}(0)
n<-seq_len(15001)
fx<-choose(n+7.3,n)*(24/25)^n/25^(8.3)


for(x in 2:15001){
    y<-1:(x-1)
    temp<-sum((17*y/(x-1)-1)*fx[y]*g[x-y])
    g[x]<-temp*16/9/(1+16/9/25^8.3)
}

ConvolveSelf<-function(n){
  convolution<-vector("numeric",2*length(n))
  for(i in 1:(length(n))){
    convolution[i]<-sum(n[1:i]*n[i:1])
  }
  for(i in 1:(length(n))){
    convolution[2*length(n)+1-i]<-sum(n[length(n)+1-(1:i)]*n[length(n)+1-(i:1)])
  }
  return(convolution)
}

g2<-ConvolveSelf(g)
g4<-ConvolveSelf(g2)
g8<-ConvolveSelf(g4)

sum(g8[20002:120000])
# remember the indices of g8 are offset by 1 so that the first index is f_S(0).
```

This also gives the probability that $S > 20000$ as 0.001810746.

[The maximum difference in estimated probabilies between these two methods is 1.541219×10^{-07} for $x = 15045$. The first method is faster, taking 7.044 seconds on my computer, while the second method takes 61.288 seconds.]