ACSC/STAT 4703, Actuarial Models II

FALL 2021
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Homework Sheet 6

Model Solutions

Basic Questions

1. *An insurer collects \$4,450,000 in earned premiums for accident year 2020. The total loss payments are \$3,831,000. Payments are subject to inflation of 7%, and policies are sold uniformly throughout the year. If the insurer's permissible loss ratio is 85%, by how much should the premium be changed for policy year 2022?*

   The loss ratio in 2020 is $\frac{3831000}{4450000} = 0.860898876404$. Without inflation, the premium should be adjusted by a factor of $\frac{0.860898876404}{0.85} = 1.01282220753$. Inflation from the start of 2020 to a random claim in accident year 2020 is

   $$\int_0^1 (1.07)^t\,dt = \left[\frac{(1.07)^t}{\log(1.07)}\right]_0^1 = \frac{0.07}{\log(1.07)} = 1.034605355$$

   Inflation from the start of 2022 to a random claim time for policy year 2022 is

   $$\begin{aligned}\int_0^1 t(1.07)^t\,dt + \int_1^2 (2-t)(1.07)^t\,dt &= \left(\frac{1.07}{\log(1.07)} - \frac{0.07}{\log(1.07)^2}\right) + 1.07\int_0^1 (1-t)(1.07)^t\,dt\\ &= \left(\frac{1.07}{\log(1.07)} - \frac{0.07}{\log(1.07)^2}\right) + 1.07\left(\int_0^1 1(1.07)^t\,dt - \int_0^1 t(1.07)^t\,dt\right)\\ &= 1.07\left(\frac{0.07}{\log(1.07}\right) - 0.07\left(\frac{1.07}{\log(1.07)} - \frac{0.07}{\log(1.07)^2}\right)\\ &= 1.07040824\end{aligned}$$

   Therefore, the premium should be adjusted by a factor $\frac{1.01282220753\times 1.07^2\times 1.07040824}{1.034605355} = 1.19970782732$

   This is an increase of 19.97%.

2. *For a certain line of insurance, an insurance company collects a total of \$6,204,000 in premiums in 2020. The company ran advertisements from May to December, and estimates that the rate of sales of new policies during that period was double the rate from the start of 2019 to the end of April 2020. Estimated incurred losses for accident year 2020 are \$3,612,000. An*

actuary is using this data to estimate rates for premium year 2024. Claims are subject to 4% inflation per year. By what percentage should premiums increase from 2020 in order to achieve a loss ratio of 0.8? [Assume that policies will be sold uniformly during the 2024 year.]

The rate of sale of policies during the last 8 months of the year was twice the usual monthly rate, so the total number of policies in force by the end of 2020 was $8 \times 2 + 4 = 20$ times the usual monthly number of policies sold. The number of policies in force at time t during 2020 is therefore proportional to

$$f(t) = \begin{cases} 1 & \text{if } t < \frac{4}{12} \\ 1 + t - \frac{4}{12} & \text{if } t \geqslant \frac{4}{12} \end{cases}$$

Thus, the earned premiums in 2020 were

$$\int_0^1 f(t)\,dt = \frac{1}{3} + \int_{\frac{1}{3}}^1 \left(t + \frac{2}{3}\right)\,dt = \frac{11}{9}$$

times the usual annual earned premiums. The premiums collected in 2020 were $\frac{20}{12} = \frac{5}{3}$ times the usual annual premiums, so the earned premimus were $\frac{11}{9} \times \frac{3}{5} \times 6204000 = \$4,549,600$. The loss ratio for 2020 is therefore $\frac{3,612,000}{4,549,600} = 0.793915948655$. To achieve a loss ratio of 0.8, without inflation, the premium needs to be decreased by a factor $\frac{0.793915948655}{0.8} = 0.992394935819$.

Now we need to account for inflation. Assuming the rate of claims is proportional to number of policies in force, the average inflation from the start of 2020 to a random claim event during the year is given by

$$\begin{aligned}
&\frac{9}{11}\left(\int_0^1 (1.04)^t\,dt + \int_{\frac{1}{3}}^1 (1.04)^t \left(t - \frac{2}{3}\right)\,dt\right)\\
=&\frac{9}{11}\left(\left[\frac{(1.04)^t}{\log(1.04)}]\right]_0^1 + (1.04)^{\frac{1}{3}} \int_0^{\frac{2}{3}} u(1.04)^u\,du\right)\\
=&\frac{9}{11}\left(\frac{(0.04)}{\log(1.04)} + (1.04)^{\frac{1}{3}}\left(\left[\frac{u(1.04)^u}{\log(1.04)}\right]_0^{\frac{2}{3}} - \int_0^{\frac{2}{3}} \frac{(1.04)^u}{\log(1.04)}\,du\right)\right)\\
=&\frac{9}{11}\left(\frac{(0.04)}{\log(1.04)} + (1.04)^{\frac{1}{3}}\left(\frac{2(1.04)^{\frac{2}{3}}}{3\log(1.04)} - \frac{(1.04)^{\frac{2}{3}} - 1}{\log(1.04)^2}\right)\right)\\
=&1.021892055
\end{aligned}$$

If policies are sold uniformly in 2023, then the average inflation from the start of 2023 to a random claim from policy year 2023 is

$$\begin{aligned}
&\int_0^1 t(1.04)^t\,dt + \int_1^2 (2-t)(1.04)^t\,dt\\
=&\left[\frac{t(1.04)^t}{\log(1.04)}\right]_0^1 - \int_0^1 \frac{1.04^t}{\log(1.04)}\,dt + (1.04)\left(\left[\frac{(1-t)(1.04)^t}{\log(1.04)}\right]_0^1 + \int_0^1 \frac{1.04^t}{\log(1.04)}\,dt\right)\\
=&\frac{(1.04)}{\log(1.04)} - \frac{0.04}{\log(1.04)^2} + (1.04)\left(-\frac{1}{\log(1.04)} + \frac{0.04}{\log(1.04)^2}\right)\\
=&\frac{(0.04)^2}{\log(1.04)^2}\\
=&1.040133323
\end{aligned}$$

Thus the premium needs to be adjusted by a factor

$$\frac{0.992394935819 \times 1.04^3 \times 1.040133323}{1.021892055} = 1.13623599929$$

That is, an increase of 13.62%.

3. *A health insurance company classifies customers as "Young", "Middle aged" and "Old". The experience from policy year 2020 is:*

Policyholder	*Current differential*	*Earned premiums (000s)*	*Loss payments (000s)*
Young	*0.53*	*3,700*	*3,420*
Middle-aged	*1*	*7,000*	*6,740*
Old	*2.64*	*6,500*	*5,590*

The base premium was \$660. Claim amounts are subject to 3% annual inflation. If the expense ratio is 25%, calculate the new premiums for each type of policyholder for policy year 2022.

We calculate the observed loss ratio and new differential for each class.

Policyholder	Old differential	Loss Ratio	New differential
Young	0.53	$\frac{3420}{3700} = 0.924324324324$	$\frac{0.53\times 0.924324324324}{0.96285714286} = 0.508789798699$
Middle-aged	1	$\frac{6740}{7000} = 0.96285714286$	$\frac{1\times 0.96285714286}{0.96285714286} = 1$
Old	2.64	$\frac{5590}{6500} = 0.86$	$\frac{2.64\times 0.86}{0.96285714286} = 2.35798219584$

With these differentials, the adjusted total earned premium is

$$\frac{3700 \times 0.508789798699}{0.53} + 7000 + \frac{6500 \times 2.35798219584}{2.64} = 16357.5667656$$

The overall loss ratio is $\frac{15750}{16357.5667656} = 0.962857142856$, so before inflation, the premium needs to be adjusted by a factor of $\frac{0.962857142856}{0.75} = 1.28380952381$ With 3% inflation for 2 years, the premium needs to be adjusted by a factor of $1.28380952381(1.03)^2 = 1.36199352381$. This is an increase of 36.62%.

The new premiums are therefore:

type	New premium
Young	$660 \times 1.36199352381 \times 0.508789798699 = \$457.36$
Middle-aged	$660 \times 1.36199352381 = \$898.92$
Old	$660 \times 1.36199352381 \times 2.35798219584 = \$2119.63$

Standard Questions

4. *A workers' compensation insurer has different premiums for manufacturing and services. Its experience for accident year 2020 is given below. There was a rate change on 18th October 2020 [292nd day of the year — note that 2020 was a leap year], which affects some of the policies.*

Policy Type	*Differential before rate change*	*Current differential*	*Earned premiums*	*Loss payments*
Manufacture	*4.31*	*4.08*	*3,206,190*	*2,938,110*
Services	*1*	*1*	*829,240*	*720,400*

Before the rate change, the base premium was $423. The current base premium is $440. Assuming that policies are sold uniformly over the year, calculate the new premimums for policy year 2023 assuming 5% annual inflation and a permissible loss ratio of 0.7.

The proportion of earned premiums under the new premium is $\frac{1}{2}\left(1-\frac{291}{366}\right)^2 = 0.0209957000806$. Therefore, the earned premiums for manufacture adjusted to the new premium is

$$3206190 \times \frac{440 \times 4.08}{0.979004299919 \times 423 \times 4.31 + 0.0209957000806 \times 440 \times 4.08} = 3158087.58665$$

. The earned premiums for services adjusted to the new premium is

$$829240 \times \frac{440}{0.979004299919 \times 423 + 0.0209957000806 \times 440} = 861839.211354$$

The loss ratios for these adjusted premiums are therefore $\frac{2938110}{3158087.58665} = 0.930344684682$ for manufacture and $\frac{720400}{861839.211354} = 0.835886776222$ for services. The new differential for manufacture is $4.08 \times \frac{0.930344684682}{0.835886776222} = 4.54105319223$. Adjusting to the new differential gives the earned premiums as $861839.211354 + 3158087.58665\frac{4.54105319223}{4.08} = 4376800.90662$. The loss ratio for this adjusted earned premium is $\frac{3658510}{4376800.90662} = 0.835886776222$. Thus to achieve a loss ratio of 0.7, the base premium should be multiplied by $\frac{0.835886776222}{0.7} = 1.19412396603$.

Using 5% annual inflation, the expected inflation from the start of 2020 to a random loss in accident year 2020 is $\int_0^1 (1.05)^t\,dt = \frac{0.05}{\log(1.05)} = 1.024796716$.

The expected inflation from the start of 2023 to a random loss in policy year 2023 is

$$\int_0^1 t(1.0t)^t\,dt + \int_1^2 (2-t)(1.0t)^t\,dt = \frac{0.05^2}{\log(1.05)^2} = 1.050208309$$

The new base premium is therefore $440\times 1.19412396603\times \frac{1.05^3\times 1.050208309}{1.024796716} = 623.315193921$. The new differential is 4.54105319223, so the premium for manufacture is $623.315193921\times 4.54105319223 = 2830.50745112$.

5. *An insurer classifies product liability insurance policyholders into food and other, and into low-risk or high-risk. It has the following data from policy year 2020:*

Number of policies

	low-risk	*high-risk*
Food	*2,734*	*1,921*
Other	*6,046*	*1,822*

loss payments

	low-risk	*high-risk*
Food	*\$4,480,200*	*\$4,343,500*
Other	*\$3,055,300*	*\$2,270,600*

The base classes are Other and low-risk, the base rate is \$506.

(a) If the differentials are 3.5 for Food and 2.4 for high-risk, calculate the new premiums which give an expense ratio of 0.25 using the loss-ratio method.

At these premiums, we have the following earned premiums and loss ratios for each class:

Class	Earned Premiums	Loss ratio
Food	$2734\times 506\times 3.5 + 1921\times 506\times 3.5\times 2.4 = \$13,006,932.4$	$\frac{8823700}{13006932.4} = 0.678384397539$
Other	$6046\times 506 + 1822\times 506\times 2.4 = \$5,271,912.8$	$\frac{5325900}{5271912.8} = 1.01024053357$
Low risk	$6046\times 506 + 2734\times 506\times 3.5 = 7,901,190.0$	$\frac{7535500}{7901190} = 0.95371709831$
High risk	$1921\times 506\times 3.5\times 2.4 + 1822\times 506\times 2.4 = 10,377,655.2$	$\frac{6614100}{10377655.2} = 0.637340504433$

This means the new differentials are $\frac{3.5\times 0.678384397539}{1.01024053357} = 2.35027729782$ for food and $\frac{2.4\times 0.637340504433}{0.95371709831} = 1.60384794752$ for high risk. Balancing back to these new differentials, the adjusted earned premiums are $506\,(2734\times 2.35027729782 + 1921\times 2.35027729782\times 1.60384794752 + 6046 + 1822\times 1.60384794752) = \$11,453,337.54$, so the overall loss ratio is $\frac{14149600}{11453337.54} = 1.23541281749$. Thus the new base premium is $\frac{506\times 1.23541281749}{0.75} = 833.491847533$. The other premiums are

	low-risk	high-risk
Food	$833.491847533\times 2.35027729782 = \$1958.94$	$833.491847533\times 2.35027729782\times 1.60384794752 = \$3141.84$
Other	\$833.49	$833.491847533\times 1.60384794752 = \$1336.79$

(b) Repeat part (a) if the differentials are 0.6 for Food and 1.1 for high-risk.

At these premiums, we have the following earned premiums and loss ratios for each class:

Class	Earned Premiums	Loss ratio
Food	$2734 \times 506 \times 0.6 + 1921 \times 506 \times 0.6 \times 1.1 = \$1,471,579.56$	$\frac{8823700}{1471579.56} = 5.99607404169$
Other	$6046 \times 506 + 1822 \times 506 \times 1.1 = \$4,073,401.2$	$\frac{5325900}{4073401.2} = 1.30748230741$
Low risk	$6046 \times 506 + 2734 \times 506 \times 0.6 = \$3,889,318.4$	$\frac{7535500}{3889318.4} = 1.93748601297$
High risk	$1921 \times 506 \times 0.6 \times 1.1 + 1822 \times 506 \times 1.1 = \$1,655,662.36$	$\frac{6614100}{1655662.36} = 3.99483624185$

This means the new differentials are $\frac{0.6\times 5.99607404169}{1.30748230741} = 2.75158172667$ for food and $\frac{1.1\times 3.99483624185}{1.93748601297} = 2.26805243322$ for high risk. Balancing back to these new differentials, the adjusted earned premiums are $506\,(2734 \times 2.75158172667 + 1921 \times 2.75158172667 \times 2.26805243322 + 6046 + 1822 \times 2.26805243322) = \$15,022,968.6866$, so the overall loss ratio is $\frac{14149600}{15022968.6866} = 0.941864440723$. Thus the new base premium is $\frac{506\times 0.941864440723}{0.75} = \$635.444542675$. The other premiums are

	low-risk	high-risk
Food	$635.444542675 \times 2.75158172667 = \$1,748.48$	$635.444542675 \times 2.75158172667 \times 2.26805243322 = \$3,965.64$
Other	$\$635.444542675$	$635.444542675 \times 2.26805243322 = \$1441.22$