ACSC/STAT 4703, Actuarial Models II

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Homework Sheet 4

Model Solutions

1. *An insurance company sells health insurance. It estimates that the standard deviation of the aggregate annual claim is \$32 and the mean is \$195.*

   (a) How many years history are needed for an individual or group to be assigned full credibility? (Use $r = 0.1$, $p = 0.9$.)

   The coefficient of variation for aggregate annual claim is $\frac{32}{195}$. For the average of n years of aggregate claims, the coefficient of variation is $\frac{32}{195\sqrt{n}}$. Using $r = 0.1$ and $p = 0.9$, the standard for full credibility is obtained by solving:

$$P\left(\left|\frac{\overline{X}-\mu}{\mu}\right| < 0.1\right) > 0.9$$
$$2\Phi\left(\frac{0.1\times 195\sqrt{n}}{32}\right) - 1 > 0.9$$
$$\frac{0.1\times 195\sqrt{n}}{32} > 1.644854$$
$$n > \left(\frac{32\times 1.644854}{0.1\times 195}\right)^2 = 7.28593755072$$

   so 8 years are needed.

   The standard net premium for this policy is \$195. An individual has claimed a total of \$133 in the last 4 years.

   (b) What is the net Credibility premium for this individual, using limited fluctuation credibility?

   The credibility of 4 years of experience is $Z = \sqrt{\frac{4}{7.28593755072}} = 0.740947220848$. The premium for this individual is therefore $0.740947220848 \times \frac{133}{4} + 0.259052779152 \times 195 = \$75.15$.

2. *A fire insurance company classifies companies as high, medium or low risk. Annual claims from high risk companies follow a Pareto distribution*

with $\alpha = 4.7$ and $\theta = 12000$. Annual claims from medium risk companies follow a Weibull distribution with $\tau = 0.6$ and $\theta = 1500$. Annual claims from low risk companies follow a gamma distribution with $\alpha = 0.7$ and $\theta = 2000$. 15% of companies are high risk, 60% are medium risk and 25% are low risk.

(a) Calculate the expectation and variance of the aggregate annual claims from a randomly chosen company.

- For a high-risk company, the expected claim is $\frac{12000}{3.7} = 3243.24324324$. The variance is $\frac{12000^2 \times 4.7}{3.7^2 \times 2.7} = 18310202.094$
- For a medium-risk company, the expected claim is $1500\Gamma\left(1 + \frac{1}{0.6}\right) = 2256.863$. The variance is $1500^2 \left(\Gamma\left(1 + \frac{2}{0.6}\right) - \Gamma\left(1 + \frac{1}{0.6}\right)^2\right) = 15742757$
- For a low-risk company, the expected claim is $2000 \times 0.7 = 1400$. The variance is $2000^2 \times 0.7 = 2800000$.

The overall expected claim amount is

$$0.15 \times 3243.24324324 + 0.6 \times 2256.863 + 0.25 \times 1400 = 2190.60428649$$

The expected squared claim amount is

$$0.15 \times (3243.24324324^2 + 18310202.094) + 0.6 \times (2256.863^2 + 15742757) + 0.25 \times (1400^2 + 2800000) = 18016036.8848$$

The variance of the claim amount is therefore $18016036.8848 - 2190.60428649^2 = 13217289.7448$.

(b) Given that a company's annual claims over the past 3 years are \$700, \$1,440 and \$320, what are the expectation and variance of the company's claims next year?

- The likelihood of these claims for a high-risk company is

$$\alpha^3 \frac{\theta^{3\alpha}}{(\theta + 700)^{\alpha+1}(\theta + 1440)^{\alpha+1}(\theta + 320)^{\alpha+1}} = 4.7^3 \frac{12000^{14.1}}{(12000 + 700)^{5.7}(12000 + 1440)^{5.7}(12000 + 320)^{5.7}}$$

- The likelihood of these claims for a medium-risk company is

$$0.6^3 \frac{700^{-0.4} \times 1440^{-0.4} \times 320^{-0.4}}{1500^{3\times 0.6}} e^{-\frac{700^{0.6}+1440^{0.6}+320^{0.6}}{1500^{0.6}}} = 2.20532452945 \times 10^{-11}$$

- The likelihood of these claims for a low-risk company is

$$\frac{700^{-0.3}1440^{-0.3}320^{-0.3}}{2000^{3\times 0.7}\Gamma(0.7)^3} e^{-\frac{700+1440+320}{2000}} = 4.377156 \times 10^{-11}$$

The posterior probabilites are therefore:

$$\frac{0.15 \times 1.96205450319 \times 10^{-11}}{0.15 \times 1.96205450319 \times 10^{-11} + 0.6 \times 2.20532452945 \times 10^{-11} + 0.25 \times 4.377156 \times 10^{-11}}$$

$$\frac{0.6 \times 2.20532452945 \times 10^{-11}}{0.15 \times 1.96205450319 \times 10^{-11} + 0.6 \times 2.20532452945 \times 10^{-11} + 0.25 \times 4.377156 \times 10^{-11}}$$

and

$$\frac{0.25 \times 4.377156 \times 10^{-11}}{0.15 \times 1.96205450319 \times 10^{-11} + 0.6 \times 2.20532452945 \times 10^{-11} + 0.25 \times 4.377156 \times 10^{-11}}$$

which gives

0.108529041709, 0.487941099394 and 0.403529858897, respectively.

This means that the expected aggregate claim is

$$0.108529041709 \times 3243.24324324 + 0.487941099394 \times 2256.863 + 0.403529858897 \times 1400 = 2018.14409708$$

The expected squared aggregate claim is

$$0.108529041709 \times (3243.24324324^2 + 18310202.094) + 0.487941099394 \times (2256.863^2 + 15742757) + 0.403529858897 \times (1400^2 + 2800000) = 1521639$$

The variance of aggregate claims is

$$15216399.5799 - 2018.14409708^2 = 11143493.9833$$

Standard Questions

3. *A home insurance company sets the standard for full credibility as 622 house-years. The book estimates are 0.07 claims per house-year for claim frequency and $4,321 per claim for claim severity.*

   The company changes the standard to 540 house-years for frequency and 86 claims for severity. For one policyholder with 11 person-years of history, this change results in the annual premium reducing from $510 to $449.11. How many claims did this policyholder make during the last 11 years?

   With the standard set as 622 house years, the credibility of this policyholder's experience is $Z = \sqrt{\frac{11}{622}} = 0.132984538424$. Therefore, in order for the premium to be \$510, the policyholder's average annual aggregate loss $\overline{X}$ must satisfy

$$\begin{aligned} Z\overline{X} + 0.07 \times 4321(1 - Z) &= 510 \\ 0.132984538424\overline{X} &= 247.753833337 \\ \overline{X} &= \$1863.02735847 \end{aligned}$$

Let the number of claims made be n. The credibility of this policyholder's experience for claim frequency is $\sqrt{\frac{11}{540}} = 0.14272480643$, so the credibility estimate for this policyholder's claim frequency is

$$0.14272480643\frac{n}{11} + 0.85727519357 \times 0.07 = 0.0129749824027n + 0.0600092635499$$

This policyholder's average claim severity is $\frac{1863.02735847 \times 11}{n} = \frac{20493.3009432}{n}$, and the credibility of this policyholder's severity is $\sqrt{\frac{n}{86}}$. This means that the credibility estimate for claim severity is

$$\sqrt{\frac{n}{86}}\frac{20493.3009432}{n} + \left(1 - \sqrt{\frac{n}{86}}\right)4321$$

Therefore the credibility premium for this policyholder is

$$(0.0129749824027n + 0.0600092635499)\left(\sqrt{\frac{n}{86}}\frac{20493.3009432}{n} + 4321\left(1 - \sqrt{\frac{n}{86}}\right)\right) = 449.11$$

$$132.611439419n^{-\frac{1}{2}} - 189.809972201 + 0.7117169329n^{\frac{1}{2}} + 56.0648989621n - 6.04563353445n^{\frac{3}{2}} = 0$$

By inspection, we see that $n = 2$ satisfies this equation.

4. *An automobile insurer classifies drivers as "low-risk" and "high-risk". It estimates that 85% of drivers are low-risk. Annual claims from low-risk drivers are modelled as following a Weibull distribution with* $\tau = 0.6$ *and* $\theta = 385$ *[mean 579.2616, variance 1037098]. Annual claims from high-risk drivers have mean \$1205 and variance 1,830,400.*

   It uses a Bayesian premium for each driver. For a particular driver with one year's experience, the net premium when modelling claims for high-risk drivers as following a gamma distribution (with $\alpha = 0.7932829$ *and* $\theta = 1519.004$ *to match the given mean and variance) is \$100 more than when modelling claims for high-risk drivers as following an inverse gamma distribution (with* $\alpha = 2.7932829$ *and* $\theta = 2160.906$ *to match the given mean and variance). What were this driver's aggregate claims for the year?*

Let p_g be the posterior probability that this driver is high-risk using the gamma distribution, and let p_i be the posterior probability that the driver is high-risk using the inverse-gamma distribution. The Bayesian premiums are $1205p_g + 579.2616(1-p_g) = 625.7384p_g + 579.2616$ for the Gamma distribution and $625.7384p_i + 579.2616$ for the inverse Gamma distribution. Since the difference between these premiums is \$100, it follows that $625.7384(p_g - p_i) = 100$, so $p_g - p_i = 0.159811192665$.

Let L_g, L_i and L_w be the likelihood of the drivers aggregate claims under the gamma, inverse gamma and Weibull distributions respectively. We have $p_g = \frac{0.15L_g}{0.15L_g+0.85L_w}$ and $p_i = \frac{0.15L_i}{0.15L_i+0.85L_w}$, so we have

$$\begin{aligned}
\frac{0.15L_g}{0.15L_g+0.85L_w} - \frac{0.15L_i}{0.15L_i+0.85L_w} &= 0.159811192665\\
0.15L_g(0.15L_i+0.85L_w) - 0.15L_i(0.15L_g+0.85L_w) &= 0.159811192665(0.15L_i+0.85L_w)(0.15L_g+0.85L_w)\\
0.1275L_w(L_g-L_i) &= 0.159811192665(0.15L_i+0.85L_w)(0.15L_g+0.85L_w)\\
0.1275L_w(L_g-L_i) &= 0.0203759270648L_w(L_g+L_i) + 0.00359575183496L_iL_g + 0.1154635867L_w^2\\
0.107124072935L_wL_g - 0.147875927065L_wL_i &= 0.00359575183496L_iL_g + 0.1154635867L_w^2 \qquad (1)
\end{aligned}$$

If the aggregate claims are X, then we have

$$\begin{aligned}
L_g &= \frac{X^{-0.2067171}e^{-\frac{X}{1519.004}}}{1519.004^{0.7932829}\Gamma(0.7932829)} = \frac{X^{-0.2067171}e^{-\frac{X}{1519.004}}}{391.5189}\\
L_i &= \frac{2160.906^{2.7932829}X^{-3.7932829}e^{-\frac{2160.906}{X}}}{\Gamma(2.7932829)} = 1237736048X^{-2.7932829}e^{-\frac{2160.906}{X}}\\
L_w &= \frac{0.6}{385^{0.6}}X^{-0.4}e^{-\left(\frac{X}{385}\right)^{0.6}} = 0.0168606693902X^{-0.4}e^{-\left(\frac{X}{385}\right)^{0.6}}
\end{aligned}$$

Substituting these into (1) and performing a grid-search

```
x<-seq_len(50000)/50
Lg<-x^{-0.2067171}*exp(-x/1519.004)/391.5189
Li<-1237736048*x^(-2.7932829)*exp(-2160.906/x)
Lw<-0.0168606693902*x^(-0.4)*exp(-(x/385)^(0.6))
plot(x,0.107124072935*Lw*Lg-0.147875927065*Lw*Li-0.00359575183496*Li*Lg+0.1154635867*Lw^2,type='l')

sum(0.107124072935*Lw*Lg-0.147875927065*Lw*Li-0.00359575183496*Li*Lg+0.1154635867*Lw^2>0)


x<-163.6+seq_len(10000)/100000


Lg<-x^{-0.2067171}*exp(-x/1519.004)/391.5189
Li<-1237736048*x^(-2.7932829)*exp(-2160.906/x)
Lw<-0.0168606693902*x^(-0.4)*exp(-(x/385)^(0.6))

plot(x,0.107124072935*Lw*Lg-0.147875927065*Lw*Li-0.00359575183496*Li*Lg+0.1154635867*Lw^2,type='l')

sum(0.107124072935*Lw*Lg-0.147875927065*Lw*Li-0.00359575183496*Li*Lg+0.1154635867*Lw^2>0)

(0.107124072935*Lw*Lg-0.147875927065*Lw*Li-0.00359575183496*Li*Lg+0.1154635867*Lw^2)[9213]
(0.107124072935*Lw*Lg-0.147875927065*Lw*Li-0.00359575183496*Li*Lg+0.1154635867*Lw^2)[9214]
x[9214]
```

gives $X = 163.6921$.

5. *An insurance company is pricing a professional liability insurance policy for a company. It has 5 years of past history for this company, and the annual claims from year i are denoted X_i. It uses the formula $\hat{X}_6 = \alpha_0 + \sum_{i=1}^{5} \alpha_i X_i$. It makes the following assumptions about the losses each year:*

   - *The expected aggregate claims was \$3052 in Year 1 and has been increasing by 4% inflation each year since then.*
   - *The coefficient of variation for aggregate claims is 2.7 in each year.*
   - *The correlation between losses in years i and j is $0.84(0.93^{|i-j|})$ if $i, j \neq 3$ and $0.62(0.93^{|i-3|})$ if $j = 3$. The change in Year 3 is to cover parental leave for one of the consultants working for the company. (Recall Corr$(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}}$)*

   Find a set of equations which can determine the values of α_i for $i = 0, 1, \ldots, 5$. [You do not need to solve these equations.]

   We use our standard equations:

$$\mathbb{E}(X_6) = \alpha_0 + \sum_{i=1}^{5} \alpha_i \mathbb{E}(X_i)$$

$$\mathrm{Cov}(X_6, X_j) = \sum_{i=1}^{5} \alpha_i \,\mathrm{Cov}(X_i, X_j)$$

From the first condition, we have $\mathbb{E}(X_i) = 3052(1.04)^{i-1}$, $\mathrm{Var}(X_i) = \left(2.7 \times 3052(1.04)^{i-1}\right)^2 = 67904192.16 \times 1.04^{(2i-2)}$, and

$$\mathrm{Cov}(X_i, X_j) = \begin{cases} 57039521.4144(0.93)^{|i-j|}(1.04)^{i+j-2} & \text{if } i, j \neq 3 \\ 42100599.1392(0.93)^{|i-3|}(1.04)^{i+1} & \text{if } j = 3 \end{cases}$$

Substituting in the numbers given, these equations become:

$$\begin{aligned}
3052(1.04)^5 &= \alpha_0 + 3052\alpha_1 + 3052(1.04)^1\alpha_2 + 3052(1.04)^2\alpha_3 + 3052(1.04)^3\alpha_4 + 3052(1.04)^4\alpha_5 \\
57039521.4144(0.93)^5(1.04)^5 &= 67904192.16\alpha_1 + 57039521.4144(0.93)(1.04)\alpha_2 + 42100599.1392(0.93)^2(1.04)^2\alpha_3 \\
&\quad + 57039521.4144(0.93)^3(1.04)^3\alpha_4 + 57039521.4144(0.93)^4(1.04)^4\alpha_5 \\
57039521.4144(0.93)^4(1.04)^6 &= 57039521.4144(0.93)(1.04)\alpha_1 + 67904192.16(1.04)^2\alpha_2 + 42100599.1392(0.93)(1.04)^3\alpha_3 \\
&\quad + 57039521.4144(0.93)^2(1.04)^4\alpha_4 + 57039521.4144(0.93)^3(1.04)^5\alpha_5 \\
42100599.1392(0.93)^3(1.04)^7 &= 42100599.1392(0.93)^2(1.04)^2\alpha_1 + 42100599.1392(0.93)(1.04)^3\alpha_2 + 67904192.16(1.04)^4\alpha_3 \\
&\quad + 42100599.1392(0.93)(1.04)^5\alpha_4 + 42100599.1392(0.93)^2(1.04)^6\alpha_5 \\
57039521.4144(0.93)^2(1.04)^8 &= 57039521.4144(0.93)^3(1.04)^3\alpha_1 + 57039521.4144(0.93)^2(1.04)^4\alpha_2 + 42100599.1392(0.93)(1.04)^5\alpha_3 \\
&\quad + 67904192.16(1.04)^6\alpha_4 + 57039521.4144(0.93)(1.04)^6\alpha_5 \\
57039521.4144(0.93)(1.04)^9 &= 57039521.4144(0.93)^4(1.04)^4\alpha_1 + 57039521.4144(0.93)^3(1.04)^5\alpha_2 + 42100599.1392(0.93)^2(1.04)^6\alpha_3 \\
&\quad + 57039521.4144(0.93)(1.04)^7\alpha_4 + 67904192.16(1.04)^8\alpha_5
\end{aligned}$$