ACSC/STAT 4703, Actuarial Models II

FALL 2022
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Homework Sheet 6

Model Solutions

Basic Questions

1. *An insurer collects \$6,820,000 in earned premiums for accident year 2021. The total loss payments are \$5,391,000. Payments are subject to inflation of 4%, and policies are sold uniformly throughout the year. If the insurer's permissible loss ratio is 75%, by how much should the premium be changed for policy year 2023?*

   The loss ratio in 2021 is $\frac{5391000}{6820000} = 0.790469208211$. Without inflation, the premium should be adjusted by a factor of $\frac{0.790469208211}{0.75} = 1.05395894428$. Inflation from the start of 2021 to a random claim in accident year 2021 is

   $$\int_0^1 (1.04)^t\,dt = \left[\frac{(1.04)^t}{\log(1.04)}\right]_0^1 = \frac{0.04}{\log(1.04)} = 1.01986926764$$

   Inflation from the start of 2023 to a random claim time for policy year 2023 is

   $$\begin{aligned}\int_0^1 t(1.04)^t\,dt + \int_1^2 (2-t)(1.04)^t\,dt &= \left(\frac{1.04}{\log(1.04)} - \frac{0.04}{\log(1.04)^2}\right) + 1.04\int_0^1 (1-t)(1.04)^t\,dt\\ &= \left(\frac{1.04}{\log(1.04)} - \frac{0.04}{\log(1.04)^2}\right) + 1.04\left(\int_0^1 1(1.04)^t\,dt - \int_0^1 t(1.04)^t\,dt\right)\\ &= 1.04\left(\frac{0.04}{\log(1.04}\right) - 0.04\left(\frac{1.04}{\log(1.04)} - \frac{0.04}{\log(1.04)^2}\right)\\ &= 1.04013332308\end{aligned}$$

   Therefore, the premium should be adjusted by a factor $\frac{1.05395894428\times 1.04^2\times 1.04013332308}{1.01986926764} = 1.1626122041$

   This is an increase of 16.26%.

2. *A liability insurance company classifies companies as "Technology", "Medical" and "Other". The experience from policy year 2021 is:*

Policyholder	*Current differential*	*Earned premiums (000s)*	*Loss payments (000s)*
Technology	*1*	*9,600*	*8,060*
Medical	*1.83*	*6,400*	*5,330*
Other	*0.47*	*7,800*	*6,170*

The base premium was \$1,530. Claim amounts are subject to 5% annual inflation. If the expense ratio is 20%, calculate the new premiums for each type of policyholder for policy year 2023.

We calculate the observed loss ratio and new differential for each class.

Policyholder	Old differential	Loss Ratio	New differential
Technology	1	$\frac{8060}{9600} = 0.839583333333$	$\frac{1\times 0.839583333333}{0.839583333333} = 1$
Medical	1.83	$\frac{5330}{6400} = 0.8328125$	$\frac{1.83\times 0.8328125}{0.839583333333} = 1.81524193548$
Other	0.47	$\frac{6170}{7800} = 0.791025641026$	$\frac{0.47\times 0.791025641026}{0.839583333333} = 0.442817331552$

With these differentials, the adjusted total earned premium is

$$9600 + \frac{6400 \times 1.81524193548}{1.83} + \frac{7800 \times 0.442817331552}{0.47} = 23297.2704715$$

The overall loss ratio is $\frac{19560}{23297.2704715} = 0.839583333332$, so before inflation, the premium needs to be adjusted by a factor of $\frac{0.839583333332}{0.8} = 1.04947916667$ With 5% inflation for 2 years, the premium needs to be adjusted by a factor of $1.04947916667(1.05)^2 = 1.15705078125$. This is an increase of 15.71%.

The new premiums are therefore:

type	New premium
Technology	$1530 \times 1.15705078125 = \$1,770.29$
Medical	$1530 \times 1.15705078125 \times 1.81524193548 = \$3,213.50$
Other	$1530 \times 1.15705078125 \times 0.442817331552 = \$783.91$

Standard Questions

3. *An auto insurer has different premiums for male and female drivers. Its experience for accident year 2021 is given below. There was a rate change on 9th July 2021 [190th day of the year], which affects some of the policies.*

Policy Type	*Differential before rate change*	*Current differential*	*Earned premiums*	*Loss payments*
Male	*1.22*	*1*	*2,036,420*	*1,643,290*
Female	*1*	*0.81*	*1,951,890*	*1,601,320*

Before the rate change, the base premium was \$629. The current base premium is \$760. [Note the change of base class.] Assuming that policies are sold uniformly over the year, calculate the new premimums for policy year 2023 assuming 6% annual inflation and a permissible loss ratio of 0.75.

The proportion of earned premiums under the new premium is $\frac{1}{2}\left(1 - \frac{190}{365}\right)^2 = 0.114937136423$. Therefore, the earned premiums for male adjusted to the new premium is

$$2036420 \times \frac{760}{0.885062863577 \times 629 \times 1.22 + 0.114937136423 \times 760} = 2019067.27525$$

The earned premiums for female adjusted to the new premium is

$$1951890\times\frac{760\times 0.81}{0.885062863577\times 629+0.114937136423\times 760\times 0.81}=1914996.63063$$

The loss ratios for these adjusted premiums are therefore $\frac{1643290}{2019067.27525}=0.813885708586$ for male and $\frac{1601320}{1914996.63063}=0.836199904682$ for female. The new differential for female is $0.81\times\frac{0.836199904682}{0.813885708586}=0.83220766214$. Adjusting to the new differential gives the earned premiums as $2019067.27525+1914996.63063\frac{0.83220766214}{0.81}=3986567.1135$. The loss ratio for this adjusted earned premium is $\frac{3244610}{3986567.1135}=0.813885708587$. Thus to achieve a loss ratio of 0.75, the base premium should be multiplied by $\frac{0.813885708587}{0.75}=1.08518094478$.

Using 6% annual inflation, the expected inflation from the start of 2021 to a random loss in accident year 2021 is $\int_0^1(1.06)^t\,dt=\frac{0.06}{\log(1.06)}=1.02970867194$. The expected inflation from the start of 2023 to a random loss in policy year 2023 is

$$\int_0^1 t(1.06)^t\,dt+\int_1^2(2-t)(1.06)^t\,dt=\frac{0.06^2}{\log(1.06)^2}=1.06029994908$$

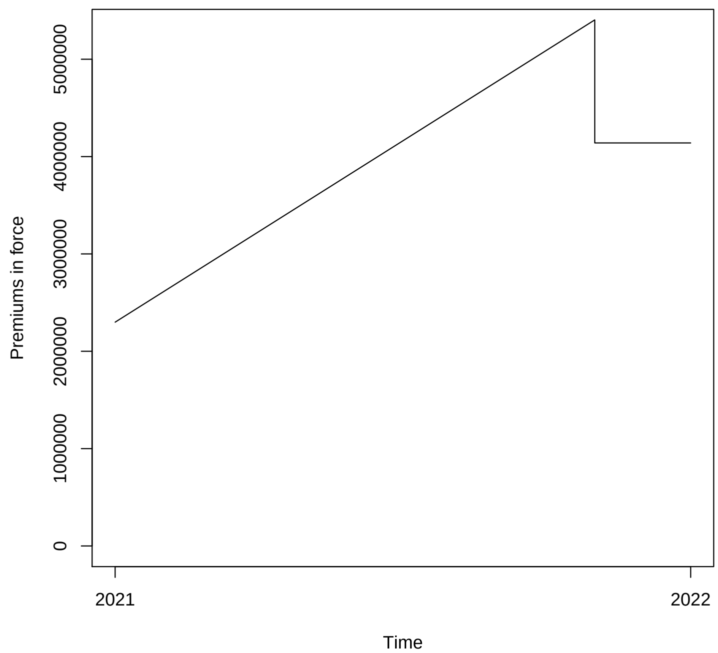
The new base premium is therefore $760\times 1.08518094478\times\frac{1.06^2\times 1.06029994908}{1.02970867194}=\$954.21$. The new differential is 0.83220766214, so the new premium for female drivers is $954.205361086\times 0.83220766214=\$794.10$.

4. *For a certain line of insurance, an insurance company collects a total of \$4,140,000 in premiums in 2021. This line of insurance was introduced at the start of October 2020, when \$1,264,000 in premiums were paid. The company assumes the rate of premiums was constant from October 2020 to December 2021. Estimated incurred losses for accident year 2021 are \$3,019,000. \$1,206,000 of these losses were in August, and for the other months of the year, losses were distributed in proportion to the number of policies in force. An actuary is using this data to estimate rates for premium year 2024. Claims are subject to 6% inflation per year. By what percentage should premiums increase from 2021 in order to achieve a loss ratio of 0.75? [Assume that policies will be sold uniformly during the 2024 year, and that claims will be follow the same pattern.]*

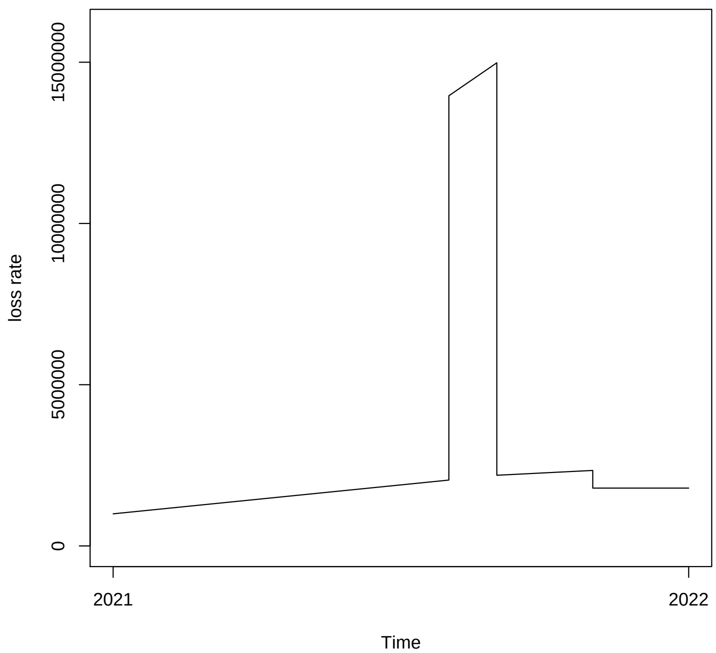
   The rate of collecting premiums in the last 3 months of 2020 is assumed to be the same as for the whole of 2021, so $4140000\times\frac{3}{12}=\$1,035,000$ would have been collected in this time period. Of this, an average of $\frac{1.5}{12}=\frac{1}{8}$ was earned in 2020, with the remaining $\frac{7}{8}$ earned in 2021. For the \$1,264,000 collected at the start of October 2020, $\frac{3}{4}$ was earned in 2021. Finally, if the \$4,140,000 collected in 2021 was uniformly distributed over the year, then half was earned in 2021. Thus the earned premiums for 2021 were

$$\frac{3}{4}\times 1264000+\frac{7}{8}\times 1035000+\frac{1}{2}\times 4140000=\$3,923,625$$

Under the assumptions given, The total premiums of policies in force in 2021 grows linearly from $1035000 + 1264000 = \$2,299,000$ at the start of the year to $2299000 + \frac{3}{4} \times 4140000 = \$5,404,000$ by the end of September. Then the total premiums of policies in force falls to \$4,140,000 and stays at this level until the end of the year.



In particular, at the start of August, the total premiums of policies in force were $2299000 + \frac{7}{9}(5404000 - 2299000) = 4714000$ at the start, and $2299000 + \frac{8}{9}(5404000 - 2299000) = 5059000$ at the end, so the average premiums in force were $\frac{4714000+5059000}{2} = 4886500$. The average premiums in force for the rest of the year were $\frac{12\times3923625-4886500}{11} = 3836090.90909$. Thus the average annual rate of losses per dollar of policy premium in force in August is $\frac{12\times1206000}{4886500} = 2.9616289778$ and for the rest of the year is $\frac{11}{12} \times \frac{1813000}{3836090.90909} = 0.433231825328$. That is, the rate of losses in August is $\frac{2.9616289778}{0.433231825328} = 6.83612976853$ times the rate of losses in other months.



For policy year 2024, the number of policies in force is uniformly distributed over the months of the year. (This is true however the policies are sold, because each policy lasts for one year.) Under the current premium with no inflation, the loss ratio would be $\frac{11}{12} \times 0.433231825328 + \frac{1}{12} \times 2.9616289778 = 0.643931588034$. Thus, without inflation, the premium should be adjusted by a factor of $\frac{0.643931588034}{0.75} = 0.858575450712$.

For inflation, the inflation from the start of 2021 to a random claim time in accident year 2021 is

$$\frac{\int_0^{\frac{7}{12}} (2299000 + 5322857.14287t)(1.06)^t\,dt + \int_{\frac{7}{12}}^{\frac{8}{12}} 6.83612976853(2299000 + 5322857.14287t)(1.06)^t\,dt + \int_{\frac{8}{12}}^{\frac{10}{12}} (2299000 + 5322857.14287t)(1.0}{\int_0^{\frac{7}{12}} (2299000 + 5322857.14287t)\,dt + \int_{\frac{7}{12}}^{\frac{8}{12}} 6.83612976853(2299000 + 5322857.14287t)\,dt + \int_{\frac{8}{12}}^{\frac{10}{12}} (2299000 + 5322857.14287t)\,dt +}$$

$$= \frac{\int_0^{\frac{10}{12}} (2299000 + 5322857.14287t)(1.06)^t\,dt + \int_{\frac{7}{12}}^{\frac{8}{12}} 6.83612976853(2299000 + 5322857.14287t)(1.06)^t\,dt + \int_{\frac{10}{12}}^{1} 4140000(1.06)^t\,dt}{\int_0^{\frac{10}{12}} (2299000 + 5322857.14287t)\,dt + \int_{\frac{7}{12}}^{\frac{8}{12}} 6.83612976853(2299000 + 5322857.14287t)\,dt + \int_{\frac{10}{12}}^{1} 4140000\,dt}$$

Recall that $\int_a^b (1.06)^t\,dt = \frac{1.06^b - 1.06^a}{\log(1.06)}$ and

$$\int_a^b t(1.06)^t = \left[t\frac{(1.06)^t}{\log(1.06)}\right]_a^b - \int_a^b \frac{(1.06)^t}{\log(1.06)}\,dt = \frac{b(1.06)^b - a(1.06)^a}{\log(1.06)} - \frac{(1.06)^b - (1.06)^a}{\log(1.06)^2}$$

This gives the following formula for average inflation

$$\frac{2299\frac{1.06^{\frac{10}{12}}-1}{\log(1.06)}+13417\frac{1.06^{\frac{8}{12}}-1.06^{\frac{7}{12}}}{\log(1.06)}+4140\frac{1.06-1.06^{\frac{10}{12}}}{\log(1.06)}+5323\left(\frac{\frac{10}{12}1.06^{\frac{10}{12}}}{\log(1.06)}-\frac{(1.06)^{\frac{10}{12}}-1}{\log(1.06)^2}\right)+31064\left(\frac{\frac{8}{12}(1.06)^{\frac{8}{12}}-\frac{7}{12}(1.06)^{\frac{7}{12}}}{\log(1.06)}-\frac{(1.06)^{\frac{8}{12}}-(1.06)^{\frac{7}{12}}}{\log(1.06)^2}\right)}{2299\times\frac{10}{12}+5323\times\frac{100}{288}+5.8361\times 2299\times\frac{1}{12}+\frac{5.8361\times 5323}{2}\left(\frac{8^2-7^2}{12^2}\right)+4140\times\frac{2}{12}}$$

$$=\frac{7437757.30986}{7190115.5756}=1.03444196851$$

The inflation from the start of 2024 to a random claim in policy year 2024 is

$$\frac{\int_0^1 t(1.06)^t\,dt+5.83612976853\int_{\frac{7}{12}}^{\frac{8}{12}}t(1.06)^t\,dt+\int_1^2(2-t)(1.06)^t\,dt+5.83612976853\int_{\frac{19}{12}}^{\frac{20}{12}}(2-t)(1.06)^t\,dt}{\int_0^1 t\,dt+5.83612976853\int_{\frac{7}{12}}^{\frac{8}{12}}t\,dt+\int_1^2(2-t)\,dt+5.83612976853\int_{\frac{19}{12}}^{\frac{20}{12}}(2-t)\,dt}$$

$$=\frac{\int_0^1 t(1.06)^t\,dt+5.83612976853\int_{\frac{7}{12}}^{\frac{8}{12}}t(1.06)^t\,dt+(1.06)\int_0^1(1-t)(1.06)^t\,dt+5.83612976853(1.06)\int_{\frac{7}{12}}^{\frac{8}{12}}(1}{1+\frac{5.83612976853}{12}}$$

$$=\frac{(1.06)\int_0^1(1.06)^t\,dt-0.06\int_0^1(1.06)^t\,dt+5.83612976853\left(1.06\int_{\frac{7}{12}}^{\frac{8}{12}}(1.06)^t\,dt-0.06\int_{\frac{7}{12}}^{\frac{8}{12}}t(1.06)^t\,dt\right)}{1.48634414738}$$

$$=\frac{\frac{0.06^2}{\log(1.06)^2}+5.83612976853\left(1.06\frac{1.06^{\frac{8}{12}}-1.06^{\frac{7}{12}}}{\log(1.06)}-0.06\left(\frac{\frac{8}{12}(1.06)^{\frac{8}{12}}-\frac{7}{12}(1.06)^{\frac{7}{12}}}{\log(1.06)}-\frac{(1.06)^{\frac{8}{12}}-(1.06)^{\frac{7}{12}}}{\log(1.06)^2}\right)\right)}{1.48634414738}$$

$$\frac{1.5760303616}{1.48634414738}=1.06034014019$$

Thus the premium needs to be adjusted by a factor $0.858575450712\times(1.06)^3\times\frac{1.06034014019}{1.03444196851}=1.04817822316$.

5. *An insurer classifies home insurance policyholders into apartment and house, and into low-risk or high-risk. It has the following data from policy year 2021:*

Number of policies

	low-risk	*high-risk*
Apartment	*8,205*	*3,052*
House	*4,631*	*11,822*

loss payments

	low-risk	*high-risk*
Apartment	*\$2,269,400*	*\$1,191,000*
House	*\$2,074,300*	*\$11,460,700*

The base classes are House and high-risk, the base rate is \$1211.

(a) If the differentials are 0.38 for Apartment and 0.67 for low-risk, calculate the new premiums which give an expense ratio of 0.2 using the loss-ratio method.

At these premiums, we have the following earned premiums and loss ratios for each class:

Class	Earned Premiums	Loss ratio
Apartment	$8205 \times 1211 \times 0.38 \times 0.67 + 3052 \times 1211 \times 0.38 = \$3,934,239.88$	$\frac{3460400}{3934239.883} = 0.879559991996$
House	$4631 \times 1211 \times 0.67 + 11822 \times 1211 = \$18,073,896.47$	$\frac{13535000}{18073896.47} = 0.748870063656$
Low risk	$8205 \times 1211 \times 0.38 \times 0.67 + 4631 \times 1211 \times 0.67 = \$6,287,224.99$	$\frac{4343700}{6287224.993} = 0.690877136548$
High risk	$3052 \times 1211 \times 0.38 + 11822 \times 1211 = \$15,720,911.36$	$\frac{12651700}{15720911.36} = 0.804768865512$

This means the new differentials are $\frac{0.67 \times 0.690877136548}{0.804768865512} = 0.575180901404$ for low-risk and $\frac{0.38 \times 0.879559991996}{0.748870063656} = 0.446316141049$ for apartment Balancing back to these new differentials, the adjusted earned premiums are

$$1211\,(8205 \times 0.446316141049 \times 0.575180901404 + 3052 \times 0.446316141049 + 4631 \times 0.575180901404 + 11822) = \$21,742,470.6196$$

so the overall loss ratio is $\frac{16995400}{21742470.6196} = 0.781668297837$. Thus the new base premium is $\frac{1211 \times 0.781668297837}{0.8} = \$1,183.25038585$. The other premiums are

	low-risk	high-risk
Apartment	$1183.25038585 \times 0.446316141049 \times 0.575180901404 = \$303.76$	$1183.25038585 \times 0.446316141049 = \$528.10$
House	$1183.25038585 \times 0.575180901404 = \$680.58$	\$1183.25

(b) What differentials for 2021 would make the new premiums before inflation \$592 for low-risk houses, and \$1381 for high-risk houses?

These premiums mean that the new differential for low-risk must be $\frac{592}{1381} = 0.42867487328$. The base premium is increased by a factor $\frac{1381}{1211} = 1.14037985136$, which means that after balancing back to the new differentials, the loss ratio must be $1.14037985136 \times 0.8 = 0.912303881088$. Therefore, the adjusted earned premiums must be $\frac{16995400}{0.912303881088} = 18629099.7466$. If the new differential for apartment is d, then this gives

$$\begin{aligned}
1211 \times 11822 + 1211 \times 0.42867487328 \times 4631 + (1211 \times 3052 + 1211 \times 0.42867487328 \times 8205)\,d &= 18629099.7466 \\
7955394.853d + 16720511.1325 &= 18629099.7466 \\
d = \frac{1908588.6141}{7955394.853} &= 0.239911236258
\end{aligned}$$

So the new differential for apartment must be 0.239911236258. Let the original differentials for apartment and low-risk be a and l respectively. The loss ratios for each class are

Class	Earned Premiums
Apartment	$8205 \times 1211al + 3052 \times 1211a = 9936255al + 3695972a$
House	$4631 \times 1211l + 11822 \times 1211 = 5608141l + 14316442$
Low risk	$8205 \times 1211al + 4631 \times 1211a = 9936255al + 5608141a$
High risk	$3052 \times 1211a + 11822 \times 1211 = 3695972a + 14316442$

For the new differential for apartment to be 0.239911236258, we must have

$$\begin{aligned}
a \times \frac{3460400}{9936255al + 3695972a} \times \frac{5608141l + 14316442}{13535000} &= 0.239911236258 \\
3460400(5608141l + 14316442) &= 0.239911236258 \times 13535000(9936255l + 3695972) \\
19406411116400l + 49540615896800 &= 32264993153900l + 12001555040300 \\
12858582037500l &= 37539060856500 \\
l &= 2.91937794906
\end{aligned}$$

For the new differential for low-risk to be 0.42867487328, we must have

$$\begin{aligned}
l \times \frac{4343700}{9936255al + 5608141l} \times \frac{3695972a + 14316442}{12651700} &= 0.42867487328 \\
4343700(3695972a + 14316442) &= 0.42867487328 \times 12651700(9936255a + 5608141) \\
16054193576400a + 62186329115400 &= 53888940109500a + 30415561443800 \\
37834746533100a &= 31770767671600 \\
a &= 0.839724607215
\end{aligned}$$

Thus if the differentials in 2021 were 0.839724607215 for apartment and 2.91937794906 for low-risk, we would get the given premiums for 2022.