ACSC/STAT 4703, Actuarial Models II

FALL 2023
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Practice Final Examination

Model Solutions

This Sample examination has more questions than the actual final, in order to cover a wider range of questions. Estimated times are provided after each question to help your preparation.

1. *An insurance company has the following data on its policies:*

Policy limit	*Losses Limited to*			
	20,000	*50,000*	*100,000*	*500,000*
20,000	*5,400,000*			
50,000	*4,590,000*	*6,070,000*		
100,000	*12,900,000*	*16,000,000*	*18,400,000*	
500,000	*9,200,000*	*11,100,000*	*13,800,000*	*16,200,000*

Use this data to calculate the ILF from \$20,000 to \$500,000 using

(a) The direct ILF estimate. [5 mins]

The direct ILF estimate is $\frac{16200000}{9200000} = 1.76086956522$.

(b) The incremental method. [5 mins]

Using the incremental method the ILFs are:

\$20,000–\$50,000	$\frac{6070000+16000000+11100000}{4590000+12900000+9200000} = 1.24278756088$
\$50,000–\$100,000	$\frac{18400000+13800000}{16000000+11100000} = 1.18819188192$
\$100,000–\$500,000	$\frac{16200000}{13800000} = 1.17391304348$

So the ILF is $1.24278756088 \times 1.18819188192 \times 1.17391304348 = 1.73348228049$.

2. *An insurance company charges a risk charge equal to the square of the average loss amount, divided by 20,000. It has the following data on a set of 1,700 claims from policies with limit \$1,000,000.*

Losses Limited to	*50,000*	*100,000*	*500,000*	*1,000,000*
Total claimed	*9,500,000*	*14,060,000*	*17,220,000*	*21,390,000*

Calculate the ILF from \$500,000 to \$1,000,000. [10 mins]

For limit \$500,000, the expected loss amount is $\frac{17220000}{1700} = 10129.4117647$, and the risk charge is $\frac{10129.4117647^2}{20000} = 5130.24913495$. The premium is therefore $10129.4117647 + 5130.24913495 = 15259.6608997$. For limit \$1,000,000, the expected loss amount is $\frac{21390000}{1700} = 12582.3529412$, and the risk charge is $\frac{12582.3529412^2}{20000} = 7915.78027685$, so the premium is $12582.3529412 + 7915.78027685 = 20498.1332181$. The ILF is therefore $\frac{20498.1332181}{15259.6608997} = 1.34328890746$.

3. *An insurer models a loss as following an inverse Weibull distribution with $\tau = 3$ and $\theta = 100$. What are the parameters c_n and d_n that make the distribution of $\frac{M_n - d_n}{c_n}$ converge, where M_n are block maxima of a block of n samples, and what is the limiting distribution? [15 mins]*

   The limiting distribution has distribution function given by

$$-\log(H(x)) = \lim_{n\to\infty} nS(c_n x + d_n)$$

   The survival function of the inverse Weibull distribution is $S(x) = 1 - e^{-\left(\frac{100}{x}\right)^3}$, so we want to find c_n and d_n such that

$$\lim_{n\to\infty} n\left(1 - e^{-\left(\frac{100}{c_n x + d_n}\right)^3}\right) = -\log(H(x))$$

   For this to work, we must have $c_n x + d_n \to \infty$, so $e^{-\left(\frac{100}{c_n x + d_n}\right)^3} \to 1 - \left(\frac{100}{c_n x + d_n}\right)^3$, which gives

$$\lim_{n\to\infty} n\left(1 - e^{-\left(\frac{100}{c_n x + d_n}\right)^3}\right) = \lim_{n\to\infty} \left(\frac{100 n^{\frac{1}{3}}}{c_n x + d_n}\right)^3$$

   Clearly, this converges if $c_n = cn^{\frac{1}{3}}$ and $d_n = dn^{\frac{1}{3}}$ for constants c and d. This gives $\log(H(x)) = \left(\frac{100}{cx+d}\right)^3$. This is a Fréchet distribution.

4. *An insurer models aggregate daily losses with a distribution in the MDA of a Gumbel distribution. Of the past 200 years, 29 years included daily losses exceeding \$100,000, and 17 years included daily losses exceeding \$500,000. What is the probability of a daily loss exceeding \$1,000,000 during the next year? [10 mins]*

   We have that $\frac{M_{365} - d_{365}}{c_{365}}$ follows a Gumbel distribution. The distribution function of this is $F(x) = e^{-e^{-x}}$. We have that $P(M_{365} < 100000) = 0.855$ and $P(M_{365} < 500000) = 0.915$. Solving $F(x) = 0.855$ gives $x = -\log(-\log(0.855)) = 1.85371693986$ and solving $F(x) = 0.915$ gives $x = -\log(-\log(0.915)) = 2.42101718504$. Thus, we have $\frac{100000 - d_{365}}{c_{365}} = 1.85371693986$ and $\frac{500000 - d_{365}}{c_{365}} = 2.42101718504$. We solve the equations

$$\begin{aligned}
1.85371693986 c_{365} + d_{365} &= 100000 \\
2.42101718504 c_{365} + d_{365} &= 500000 \\
0.56730024518 c_{365} &= 400000 \\
c_{365} &= 705094.001631 \\
d_{365} &= -1207044.69502
\end{aligned}$$

   Thus, the probability that the next year includes a daily loss exceeding \$1,000,000 is

$$\begin{aligned}
P(M_{365} > 1000000) &= P\left(\frac{M_{365} - d_{365}}{c_{365}} > \frac{1000000 + 1207044.69502}{705094.001631}\right) \\
&= 1 - F(3.13014249152) \\
&= 1 - e^{-e^{-3.13014249152}} \\
&= 0.042769986842
\end{aligned}$$

5. *A reinsurer offers an excess-of-loss reinsurance contract on a portfolio with attachment point \$5,000,000 and a policy limit of \$5,000,000. The aggregate loss distribution is estimated to lie in the MDA of a Fréchet distribution with* $\xi = 0.2$. *The reinsurer estimates that the probability of paying a claim is 0.06 and the probability that the policy limit is reached is 0.0002. What is the expected payment on the contract. [10 mins]*

   Since the distribution is in the MDA of a Fréchet distribution, the excess loss distribution converges to a Pareto distribution with $\alpha = 5$. The probability of a payment is 0.06 and the probability of the policy limit being exceeded is 0.0002. Thus, the probability that the excess loss distribution exceeds 5,000,000 is $\frac{0.0002}{0.06} = \frac{1}{300}$. Thus the scale parameter of the Pareto distribution satisfies

$$\begin{aligned}
\left(\frac{\theta}{\theta + 5000000}\right)^5 &= \frac{1}{300} \\
1 + \frac{5000000}{\theta} &= 300^{\frac{1}{5}} \\
\theta &= \frac{5000000}{300^{\frac{1}{5}} - 1} \\
&= 2348371.91384
\end{aligned}$$

   Conditional on a claim being made, the expected payment is

$$\begin{aligned}
\int_0^{5000000} S_{5000000}(x)\,dx &= \int_0^{5000000} \left(\frac{\theta}{\theta + x}\right)^5 dx \\
&= \int_\theta^{\theta+5000000} \theta^5 u^{-5}\,du \\
&= \theta^5 \left[-\frac{u^{-4}}{4}\right]_\theta^{\theta+5000000} \\
&= \frac{1}{4}\left(\theta - \frac{\theta^5}{(\theta + 5000000)^4}\right) \\
&= 580969.335198
\end{aligned}$$

6. *An insurer models claims as following a distribution in the MDA of a GEV distribution with* $\xi = -2.5$. *They find that the probability of a claim exceding \$1,000,000 is 0.04 and the probability of a claim exceeding \$2,000,000 is 0.008. What is the maximum possible claim under this model?*

   Under the GPD approximation, the excess claim above \$1,000,000 follows a GPD distribution with $\xi = -2.5$. The survival function of this distribution is $S(x) = \left(1 + \frac{\xi}{\beta}x\right)^{-\frac{1}{\xi}}$. We are given that $S(1,000,000) = \frac{0.008}{0.04} = 0.2$, which gives

$$\left(1-\frac{2.5\times 1000000}{\beta}\right)^{0.4}=0.2$$
$$\frac{2.5\times 1000000}{\beta}=1-0.2^{2.5}=0.98211145618$$
$$\beta=\frac{2.5\times 1000000}{0.98211145618}=2545535.93105$$

The maximum value of excess loss is $-\frac{\beta}{\xi}=\frac{2545535.93105}{2.5}=1018214.37242$. Thus, the maximum claim amount is $1000000+1018214.37242=\$2,018,214.37242$.

7. *An actuary is reviewing a sample of 75,060 observations that he believes comes from the MDA of a Fréchet distribution. He uses the Hill estimator to estimate ξ. He calculates $\hat{\alpha}_j$ for a range of different thresholds j:*

j	$\hat{\alpha}_j$
73,000	*2.842*
74,000	*3.692*

Given that $x_{(73000)}=12493$, which of the following is a possible value for $x_{(74000)}$? Justify your answer.

(i) 12986

(ii) 16986

(iii) 24986

(iv) 29986

[15 mins]

The Hill estimator is given by

$$\hat{\xi}=\frac{1}{N-j+1}\sum_{k=j+1}^{N}\log(x_{(k)})-\log(x_{(j)})$$

Thus, we have that

$$\frac{1}{2061}\sum_{k=73001}^{N}\log(x_{(k)})-\log(12493)=\frac{1}{2.842}=0.351864883885$$

and

$$\frac{1}{1061}\sum_{k=74001}^{N}\log(x_{(k)})-\log(x_{74000})=\frac{1}{3.692}=0.270855904659$$

Thus

$$\sum_{k=73001}^{75060}\log(x_{(k)})=2061\left(0.351864883885+\frac{2060}{2061}\log(12493)\right)=20157.0164845$$

and

$$\frac{1}{2060}\sum_{k=73001}^{75060}\log(x_{(k)})=9.7849594585$$

so

$$\sum_{k=74001}^{75060} \log(x_{(k)}) \geqslant 1060 \times 9.7849594585 = 10372.057026$$

and thus

$$\log(x_{74000}) = \frac{\sum_{k=74001}^{N} \log(x_{(k)})}{1060} - \frac{1061}{1060} \times 0.270855904659 \geqslant \frac{10372.057026}{1060} - 0.270855904659 = 9.51410355383$$

giving

$$x_{74000} \geqslant 13549.4814178$$

On the other hand, for $73000 < j < 74000$, we have $x_{(j)} > x_{(73000)} = 12493$ and for $74000 \leqslant j$, we have $x_{(j)} \geqslant x_{(74000)}$. Thus,

$$20157.0164845 = \sum_{k=73001}^{75060} \log(x_{(k)}) \geqslant 1000 \log(12493) + 1060 \log(x_{(74000)}) = 9432.92376643 + 1060 \log(x_{(74000)})$$

Thus,

$$\log(x_{(74000)}) \leqslant \frac{20157.0164845 - 9432.92376643}{1060} = 10.117068602$$

giving $x_{(74000)} \leqslant 24762.0768371$

Thus (ii) $x_{(74000)} = 16986$ is the only possible answer.

8. *Loss amounts follow an exponential distribution with $\theta = 3,000$. The distribution of the number of losses is given in the following table:*

Number of Losses	*Probability*
0	*0.64*
1	*0.28*
2	*0.08*

Assume all losses are independent and independent of the number of losses. The insurance company buys excess-of-loss reinsurance on the part of the loss above \$7,500. Calculate the expected payment for this excess-of-loss reinsurance. [15 mins]

If the number of losses is n, then the aggregate loss follows a gamma distribution with $\alpha = n$ and $\theta = 3000$. The expected payment on the excess-of-loss insurance is therefore

$$\begin{aligned}
&\int_{7500}^{\infty} (x - 7500) \frac{x^{n-1} e^{-\frac{x}{3000}}}{(n-1)! 3000^n} \, dx \\
&= \int_{7500}^{\infty} \frac{x^n e^{-\frac{x}{3000}}}{(n-1)! 3000^n} \, dx - 7500 \int_{3000}^{\infty} \frac{x^{n-1} e^{-\frac{x}{3000}}}{(n-1)! 3000^n} \, dx \\
&= \int_{2.5}^{\infty} \frac{3000 n u^n e^{-u}}{n!} \, du - 7500 \int_{2.5}^{\infty} \frac{u^{n-1} e^{-u}}{(n-1)!} \, du
\end{aligned}$$

This gives the following expected payments on the excess-of-loss reinsurance:

Number of Losses	Probability	Expected payment on excess-of-loss	product
0	0.64	0	0
1	0.28	$3000 \times 1 \times 0.2872975 - 7500 \times 0.0820850 = 246.255$	68.9514
2	0.08	$3000 \times 2 \times 0.5438131 - 7500 \times 0.2872975 = 1108.14735$	88.651788

The total expected payment on the excess-of-loss reinsurance is therefore $68.9514 + 88.651788 = \$157.60$.

9. *Claim frequency follows a negative binomial distribution with $r = 0.6$ and $\beta = 0.6$. Claim severity (in thousands) has the following distribution:*

Severity	*Probability*
0	*0.352*
1	*0.384*
2	*0.217*
3 or more	*0.047*

Use the recursive method to calculate the exact probability that aggregate claims are at least 3. [15 mins]

For the negative binomial distribution, we have $a = \frac{\beta}{1+\beta} = \frac{0.6}{1.6} = 0.375$ and $b = \frac{(r-1)\beta}{1+\beta} = -0.4 \times 0.375 = -0.15$, so the recursive formula

$$f_S(x) = \frac{(p_1 - (a+b)p_0)f_X(x) + \sum_{i=1}^{x}\left(a + \frac{bi}{x}\right) f_X(i) f_S(x-i)}{1 - af_X(0)}$$

becomes

$$f_S(x) = \frac{\sum_{i=1}^{x} 0.375\left(1 - 0.4\frac{i}{x}\right) f_X(i) f_S(x-i)}{1 - 0.375 \times 0.352} = 0.43202764977 \sum_{i=1}^{x}\left(1 - 0.4\frac{i}{x}\right) f_X(i) f_S(x-i)$$

We calculate

$$f_S(0) = P_S(0) = P_N(P_X(0)) = P_N(f_X(0)) = (1 + 0.6 \times (1 - 0.352))^{-0.6} = 0.821138046807$$

We now use the recurrence:

$$f_S(1) = 0.43202764977 \times 0.6 \times 0.384 \times 0.821138046807 = 0.0817354000508$$
$$f_S(2) = 0.43202764977\,(0.8 \times 0.384 \times 0.0817354000508 + 0.6 \times 0.217 \times 0.821138046807) = 0.0570368470291$$

The probability that the aggregate payments are at least 3 is therefore $1 - 0.821138046807 - 0.0817354000508 - 0.0570368470291 = 0.0400897061131$

10. *Using an arithmetic distribution ($h = 1$) to approximate an inverse Pareto distribution distribution with $\tau = 3$ and $\theta = 6$, calculate the probability that the value is between 3.5 and 6.5, for the approximation using:*

    (a) The method of rounding. [10 mins]

The method of rounding preserves this probability, since it assigns all values between 3.5 and 4.5 to 4, etc. Therefore this probability is $\left(\frac{6.5}{12.5}\right)^3 - \left(\frac{3.5}{9.5}\right)^3 = 0.0906007103075$.

(b) The method of local moment matching, matching 1 moment on each interval. [15 mins]

Using local moment matching, the probabilities of the intervals $[4, 5]$ and $[5, 6]$ are preserved, so the probability of these intervals is $\left(\frac{6}{12}\right)^3 - \left(\frac{4}{10}\right)^3 = 0.061$

For the interval $[3, 4]$, the probability of this interval is $\left(\frac{4}{10}\right)^3 - \left(\frac{3}{9}\right)^3 = 0.026962962963$ while the conditional mean times this probability is

$$\begin{aligned}
\int_3^4 x\frac{\tau\theta x^{\tau-1}}{(x+\theta)^{\tau+1}}\,dx &= 18\int_3^4 \frac{x^3}{(x+6)^4}\,dx\\
&= 18\int_9^{10} \frac{(u-6)^3}{u^4}\,du\\
&= 18\int_9^{10} u^{-1} - 18u^{-2} + 108u^{-3} - 216u^{-4}\,du\\
&= 18\left[\log(u) + 18u^{-1} - 54u^{-2} + 72u^{-3}\right]_9^{10}\\
&= 18\left(\log(10) - \log(9) - \frac{18}{9} + \frac{18}{10} + \frac{54}{9^2} - \frac{54}{10^2} - \frac{72}{9^3} + \frac{72}{10^3}\right) \qquad = 0.0947115039276
\end{aligned}$$

We are now trying to solve for p_3 and p_4 such that

$$\begin{aligned}
p_3 + p_4 &= 0.026962962963\\
3p_3 + 4p_4 &= 0.0947115039276\\
p_4 &= 0.0947115039276 - 3 \times 0.026962962963 = 0.0138226150386
\end{aligned}$$

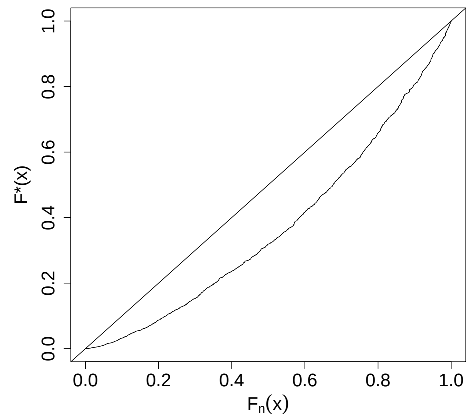
For the interval $[6, 7]$, the probability of this interval is $\left(\frac{7}{13}\right)^3 - \left(\frac{6}{12}\right)^3 = 0.0311219845244$, while the conditional mean times this probability is

$$18\left(\log(13) - \log(12) - \frac{18}{12} + \frac{18}{13} + \frac{54}{12^2} - \frac{54}{13^2} - \frac{72}{12^3} + \frac{72}{13^3}\right) = 0.202261683065$$

$$\begin{aligned}
p_6 + p_7 &= 0.0311219845244\\
6p_6 + 7p_7 &= 0.202261683065\\
p_6 &= 7 \times 0.0311219845244 - 0.202261683065 = 0.015592208606
\end{aligned}$$

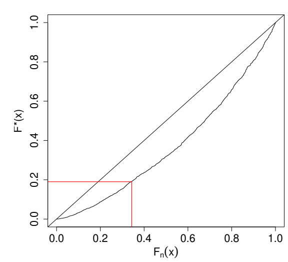
So the probability of the interval $[3.5, 6.5]$ is therefore $0.061 + 0.0138226150386 + 0.015592208606 = 0.0904148236446$.

11. *An actuary is reviewing a sample of 2015 past claims, which she believes come from a Weibull distribution with $\tau = 0.6$ and a value of θ estimated from a previous dataset. She constructs the following p-p plot to compare the sample to this distribution:*



(a) The sample included 685 points less than 1,200. What was the value of θ used in the plot? [5 mins.]

Since there are 685 points less than 1,200, we have $F_n(1200) = \frac{685}{2015} = 0.339950372208$. so we look for the point on the graph with $F_n(x) = 0.339950372208$.



We see that the corresponding value of $F^*(1200)$ is approximately 0.19. For the Weibull distribution, we have

$F^*(1200) = 1 - e^{-\left(\frac{1200}{\theta}\right)^{0.6}}$, so we solve

$$\begin{aligned}
1 - e^{-\left(\frac{1200}{\theta}\right)^{0.6}} &\approx 0.19 \\
e^{-\left(\frac{1200}{\theta}\right)^{0.6}} &\approx 0.81 \\
\left(\frac{1200}{\theta}\right)^{0.6} &\approx 0.210721031316 \\
\frac{\theta}{1200} &\approx 2.54551271147 \\
\theta &\approx 3054.61525376
\end{aligned}$$

(b) Which of the following statements best describes the fit of the Weibull distribution to the data: [5 mins.]

(i) The Weibull distribution assigns too much probability to high values and too little probability to low values.

(ii) The Weibull distribution assigns too much probability to low values and too little probability to high values.

(iii) The Weibull distribution assigns too much probability to tail values and too little probability to central values.

(iv) The Weibull distribution assigns too much probability to central values and too little probability to tail values.

The p-p plot is all below the line $F_n(x) = F^*(x)$. This means that $F_n(x) > F^*(x)$ for all x, so the model assigns too little probability to low values and too much probability to high values.

12. *A worker's compensation insurance company classifies workplaces as "safe" or "hazardous". Claims from hazardous workplaces follow a Gamma distribution with $\alpha = 0.1021749$, $\theta = 1066798$ (mean \$109,000 and standard deviation \$341,000). Claims from safe workplaces follow a Gamma distribution with $\alpha = 0.01209244$, $\theta = 2646281$ (mean \$32,000 and standard deviation \$261,000). 94% of workplaces are classified as safe.*

[You may need the following values:

$$\begin{aligned}
\Gamma(0.01209244) &= 82.13091 \\
\Gamma(0.1021749) &= 9.302457
\end{aligned}$$

]

(a) Calculate the expectation and variance of claim size for a claim from a randomly chosen workplace. [5 mins.]

The expectation is $0.94 \times 32000 + 0.06 \times 109000 = \$36,620$. The variance is $(109000 - 32000)^2 \times 0.94 \times 0.06 + 0.94 \times 261000^2 + 0.06 \times 341000^2 = 71,345,000,000$.

(b) The last 2 claims from a particular workplace are \$488,200 and \$17,400. Calculate the expectation and variance for the next claim size from this workplace. [10 mins.]

If the workplace is safe, the likelihood of these claim sizes is

$$\left(\frac{488200^{-0.98790756}e^{-\frac{488200}{2646281}}}{2646281^{0.01209244}\Gamma(0.01209244)}\right)\left(\frac{17400^{-0.98790756}e^{-\frac{17400}{2646281}}}{2646281^{0.01209244}\Gamma(0.01209244)}\right) = 1.32923 \times 10^{-14}$$

If the workplace is hazardous, the likelihood of these claim sizes is

$$\left(\frac{488200^{-0.8978251}e^{-\frac{488200}{1066798}}}{1066798^{0.1021749}\Gamma(0.1021749)}\right)\left(\frac{17400^{-0.8978251}e^{-\frac{17400}{1066798}}}{1066798^{0.1021749}\Gamma(0.1021749)}\right) = 5.134517\times 10^{-13}$$

The posterior probability that the workplace is safe is therefore $\frac{0.94\times 1.32923\times 10^{-14}}{0.94\times 1.32923\times 10^{-14}+0.06\times 5.134517\times 10^{-13}} = 0.2885502$, so the expectation is $0.2885502\times 32000 + 0.7114498\times 109000 = \$86,781.63$.

The variance is $77000^2\times 0.2885502\times 0.7114498 + 0.2885502\times 261000^2 + 0.7114498\times 341000^2 = 103,601,580,743$.

13. *item An insurance company sets the book pure premium for its home insurance at \$791. The expected process variance is 6,362,000 and the variance of hypothetical means is 341,200. If an individual has no claims over the last 8 years, calculate the credibility premium for this individual's next year's insurance using the Bühlmann model. [5 mins.]*

    The credibility is $Z = \frac{8}{8+\frac{6362000}{341200}} = 0.3002332$. Therefore the premium is $0.6997668\times 791 = \$553.52$.

14. *An insurance company is reviewing the premium for an individual with the following past claim history:*

Year	*1*	*2*	*3*	*4*	*5*
Exposure	*0.2*	*1*	*1*	*0.4*	*0.8*
Aggregate claims	*0*	*\$2,592*	*0*	*\$147*	*\$1,320*

    The usual premium per unit of exposure is \$2,700. The expected process variance is 123045 and the variance of hypothetical means is 36403 (both per unit of exposure). Calculate the credibility premium for this individual if she has 0.6 units of exposure in year 6. [10 mins.]

    The credibility of the policyholder's experience is $\frac{3.4}{3.4+\frac{123045}{36403}} = 0.5014691$. The policyholder's aggregate claims were \$4,059, so average claims per unit of exposure are $\frac{4059}{3.4} = \$1,193.53$. The credibility premium per unit of exposure is therefore $0.5014691\times 1193.53 + 0.4985309\times 2700 = \$1,944.70$. This is for a whole unit of exposure. Since the policyholder has 0.6 units of exposure, the credibility premium is $0.6\times 1944.70 = \$1,166.82$.

15. *An insurance company has 3 years of past history on a homeowner, denoted X_1, X_2, X_3. Because the individual moved house at the end of the second year, the third year has a different mean and variance, and is not as correlated with the other two years. It has the following*

$$\begin{aligned}
\mathbb{E}(X_1) &= 1,322 & \mathrm{Var}(X_1) &= 226,000\\
\mathbb{E}(X_2) &= 1,322 & \mathrm{Var}(X_2) &= 226,000\\
\mathbb{E}(X_3) &= 4,081 & \mathrm{Var}(X_3) &= 1,108,000\\
\mathbb{E}(X_4) &= 4,081 & \mathrm{Var}(X_4) &= 1,108,000\\
\mathrm{Cov}(X_1,X_2) &= 214 & \mathrm{Cov}(X_1,X_3) &= 181\\
\mathrm{Cov}(X_2,X_3) &= 181 & \mathrm{Cov}(X_1,X_4) &= 181\\
\mathrm{Cov}(X_2,X_4) &= 181 & \mathrm{Cov}(X_3,X_4) &= 861
\end{aligned}$$

    It uses a formula $\hat{X}_4 = \alpha_0 + \alpha_1 X_1 + \alpha_2 X_2 + \alpha_3 X_3$ to calculate the credibility premium in the fourth year. Calculate the values of α_0, α_1, α_2 and α_3. [15 mins.]

The company needs to choose α_0, α_1, α_2 and α_3 to satisfy:

$$
\begin{aligned}
\mathbb{E}(X_4) &= \alpha_0 + \alpha_1\mathbb{E}(X_1) + \alpha_2\mathbb{E}(X_2) + \alpha_3\mathbb{E}(X_3) \\
\mathrm{Cov}(X_4, X_1) &= \alpha_1 \mathrm{Var}(X_1) + \alpha_2 \mathrm{Cov}(X_2, X_1) + \alpha_3 \mathrm{Cov}(X_3, X_1) \\
\mathrm{Cov}(X_4, X_2) &= \alpha_1 \mathrm{Cov}(X_1, X_2) + \alpha_2 \mathrm{Var}(X_2) + \alpha_3 \mathrm{Cov}(X_3, X_2) \\
\mathrm{Cov}(X_4, X_1) &= \alpha_1 \mathrm{Cov}(X_1, X_3) + \alpha_2 \mathrm{Cov}(X_2, X_3) + \alpha_3 \mathrm{Var}(X_3)
\end{aligned}
$$

Substituting the values gives:

$$
\begin{aligned}
4081 &= \alpha_0 + 1322\alpha_1 + 1322\alpha_2 + 4081\alpha_3 \\
181 &= 226000\alpha_1 + 214\alpha_2 + 181\alpha_3 \\
181 &= 214\alpha_1 + 226000\alpha_2 + 181\alpha_3 \\
861 &= 181\alpha_1 + 181\alpha_2 + 1108000\alpha_3
\end{aligned}
$$

By symmetry, we see that α_1 and α_2 are equal. This gives

$$
\begin{aligned}
181 &= 226214\alpha_1 + 181\alpha_3 \\
861 &= 362\alpha_1 + 1108000\alpha_3 \\
226214 \times 861 - 362 \times 181 &= (226214 \times 1108000 + 362 \times 181)\alpha_3 \\
\alpha_3 &= \frac{194704732}{250,645,046,478} = 0.0007768146 \\
\alpha_1 &= \frac{181 - 181 \times 0.0007768146}{226214} = 0.0007995058 \\
\alpha_0 &= 4081 - 1322 \times 2 \times 0.0007995058 - 4081 \times 0.0007768146 = 4075.716
\end{aligned}
$$

The values are:

$$
\begin{aligned}
\alpha_0 &= 4075.716 \\
\alpha_1 &= 0.0007995058 \\
\alpha_2 &= 0.0007995058 \\
\alpha_3 &= 0.0007768146
\end{aligned}
$$

16. *An insurance company has the following previous data on aggregate claims:*

Policyholder	*Year 1*	*Year 2*	*Year 3*	*Year 4*	*Mean*	*Variance*
1	*1,210*	*246*	*459*	*1,461*	*944.00*	*340158.00*
2	*0*	*0*	*0*	*0*	*0.00*	*0.00*
3	*0*	*2,185*	*0*	*0*	*548.25*	*1202312.25*
4	*809*	*0*	*0*	*1,725*	*633.50*	*674939.00*
5	*0*	*0*	*0*	*0*	*0.00*	*0.00*

Calculate the Bühlmann credibility premium for policyholder 3 in Year 5. [15 mins.]

The expected process variance is $\frac{1}{5}(340158 + 0 + 1202312.25 + 674939 + 0) = 443421.85$. The population mean is $\frac{944+0+548.25+633.50+0}{5} = 405.15$.

total variance of estimated means is $\frac{(944-405.15)^2+(-405.15)^2+(548.25-405.15)^2+(633.50-405.15)^2+(-405.15)^2}{4} = 172318.425$. The variance of hypothetical means is therefore $172318.425 - \frac{443421.85}{4} = 61462.96$. The credibility of 4 years of experience is therefore $\frac{4}{4+\frac{443421.85}{61462.96}} = 0.3566825$. The premium for policyholder 3 is therefore $0.3566825 \times 548.25 + 0.6433175 \times 405.15 = \$456.19$.

17. *An insurance company collects the following claim frequency data for 7,000 customers insured for the past 3 years:*

No. of claims	*Frequency*
0	*1,494*
1	*2,460*
2	*1,810*
3	*827*
4	*302*
5	*72*
6	*31*
7	*3*
8	*1*
> 8	*0*

It assumes that the number of claims an individual makes in a year follows a Poisson distribution with parameter Λ, which may vary between individuals.

Find the credibility estimate for the expected number of claims per year for an individual who has made 4 claims in the past 3 years. [15 mins.]

The total number of claims in the past 3 years was $1\times2460+2\times1810+3\times827+4\times302+5\times72+6\times31+7\times3+8\times1 = 10,344$. The total number of policyholders is $1491 + 2461 + 1810 + 831 + 302 + 72 + 30 + 2 + 1 = 7,000$. The average number of claims per policyholder per year is therefore $\frac{10344}{21000} = 0.492571428571$. This is also the expected process variance. The variance of estimated means is

$$\frac{1}{6999}\left(1493\times 0.492571428571^2+2460\left(\frac{1}{3}-0.492571428571\right)^2+1810\left(\frac{2}{3}-0.492571428571\right)^2\right.$$
$$+827(1-0.492571428571)+307\left(\frac{4}{3}-0.492571428571\right)^2+72\left(\frac{5}{3}-0.492571428571\right)^2$$
$$\left.+31(2-0.492571428571)^2+3\left(\frac{7}{3}-0.492571428571\right)^2+\left(\frac{8}{3}-0.492571428571\right)^2\right)$$
$$=0.185843829802$$

The variance due to the Poisson sampling is $\frac{0.492571428571}{3}=0.16419047619$. Therefore, the variance of hypothetical means is $0.185843829802-0.16419047619=0.021653353612$. The credibility of 3 year's experience is $\frac{3}{3+\frac{0.492571428571}{0.021653353612}}=0.116513707423$. The expected number of claims is therefore $0.116513707423\times\frac{4}{3}+0.883486292577\times 0.492571428571=0.590531715155$.

18. *The following table shows the cumulative losses (in thousands) on claims from one line of business of an insurance company over the past 4 years.*

Accident year	*Development year* 0	*1*	*2*	*3*
2019	*890*	*3372*	*4563*	*4823*
2020	*1307*	*2653*	*3453*	
2021	*2742*	*6632*		
2022	*1224*			

The earned premiums in each year are given in the following table:

Year	*Earned Premiums (000's)*
2019	*5398*
2020	*6503*
2021	*8152*
2022	*7350*

Assume that payments for Accident Year 2019 have been finalised.

Calculate the total outstanding reserves using per-premium losses using:

(a) The chain-ladder method.

We first compute the per-premium cumulative losses:

Accident year	Development year 0	1	2	3
2019	0.164875879956	0.624675805854	0.845313078918	0.893479066321
2020	0.200984161156	0.407965554360	0.530985698908	
2021	0.336359175662	0.813542688911		
2022	0.166530612245			

We calculate the following loss development factors:

Development year	Loss Development Factor
1/0	$\frac{0.624675805854+0.40796555436+0.813542688911}{0.164875879956+0.200984161156+0.336359175662} = 2.62907081581$
2/1	$\frac{0.845313078918+0.530985698908}{0.624675805854+0.40796555436} = 1.33279455081$
3/2	$\frac{0.893479066321}{0.845313078918} = 1.05698005698$

Using the chain-ladder method, the estimated ultimate per-premium losses for each accident year are:

Accident year	Estimated Per-Premium Ultimate Losses	Estimated total losses
2019	0.893479	$0.893479 \times 5398 = 4823$
2020	$0.530985698908 \times 1.05698005698 = 0.561241$	$0.561241 \times 6503 = 3650$
2021	$0.813542688911 \times 1.33279455081 \times 1.05698005698 = 1.146068$	$1.146068 \times 8152 = 9343$
2022	$0.166530612245 \times 2.62907081581 \times 1.33279455081 \times 1.05698005698 = 0.616774$	$0.616774 \times 7350 = 4533$
Total		22349

The total paid so far is $4823 + 3453 + 6632 + 1224 = 16132$, so the remaining reserves are $22349 - 16132 = \$6,217$.

(b) The Bornhuetter-Fergusson method. The expected loss ratio is 0.79 and the

[15 mins]

Under the Bornhuetter-Fergusson method, the proportion of total payments in each year is given by:

Dev. Year	Proportion of total payments
0	$\frac{1}{1.05698005698\times1.33279455081\times2.62907081581} = 0.270002455704$
1	$\frac{1}{1.05698005698\times1.33279455081} - \frac{1}{1.05698005698\times1.33279455081\times2.62907081581} = 0.439853120783$
2	$\frac{1}{1.05698005698} - \frac{1}{1.05698005698\times1.33279455081} = 0.236236067718$
3	$1 - \frac{1}{1.05698005698} = 0.053908355795$

This gives us:

Accident Year	Expected total losses	Expected outstanding claims
2020	5137.37	$5137.37 \times 0.053908355795 = 276.94716981$
2021	4860.08	$4860.08 \times (0.236236067718 + 0.053908355795) = 1410.12510983$
2022	5806.50	$5806.50 \times (0.439853120783 + 0.236236067718 + 0.053908355795) = 4238.73074095$
Total		5925.80302059

(c) The Bühlmann-Straub Credibility method for per-premium losses.

In parts (a) and (b), we already calculated the following per-premium values:

Development Year j	γ_j	β_j	Accident Year i	$\hat{C}_{i,J}$
0	0.270002455704	0.270002455704	2019	0.893479
1	0.439853120783	0.709855576487	2020	0.561241
2	0.236236067718	0.946091644205	2021	1.146068
3	0.053908355795	1	2022	0.616774

We substitute these into the formulae

$$\overline{C} = \frac{\sum_{i=0}^{I} C_{i,I-i}}{\sum_{i=0}^{I} \hat{\beta}_{I-i}}$$
$$\hat{v} = \frac{1}{I}\sum_{i=0}^{I-1}\frac{1}{I-i}\sum_{j=0}^{I-i}\hat{\gamma}_j\left(\frac{X_{ij}}{\hat{\gamma}_j} - \hat{C_{i,J}}\right)^2$$
$$\hat{a} = \frac{\sum_{i=0}^{I}\hat{\beta}_{I-i}\left(\hat{C}_{i,J} - \overline{C}\right)^2 - I\hat{v}}{\sum_{i=0}^{I}\hat{\beta}_{I-i} - \frac{1}{\sum_{i=0}^{I}\hat{\beta}_{I-i}}\sum_{i=0}^{I}\hat{\beta}_{I-i}^2}$$

to get

$$\overline{C} = \frac{0.893479066321 + 0.530985698908 + 0.813542688911 + 0.166530612245}{0.270002455704 + 0.709855576487 + 0.946091644205 + 1} = 0.821797478536$$
$$\hat{v} = \frac{1}{3}\left(\frac{0.2700\left(\frac{0.1649}{0.2700} - 0.8935\right)^2 + 0.4399\left(\frac{0.4598}{0.4399} - 0.8935\right)^2 + 0.2362\left(\frac{0.2206}{0.2362} - 0.8935\right)^2 + 0.0539\left(\frac{0.0482}{0.0539} - 0.8935\right)^2}{4}\right.$$
$$+ \frac{0.2700\left(\frac{0.2010}{0.2700} - 0.5612\right)^2 + 0.4399\left(\frac{0.2070}{0.4399} - 0.5612\right)^2 + 0.2362\left(\frac{0.1230}{0.2362} - 0.5612\right)^2}{3}$$
$$\left.+ \frac{0.2700\left(\frac{0.3364}{0.2700} - 1.1461\right)^2 + 0.4399\left(\frac{0.4772}{0.4399} - 1.1461\right)^2}{2}\right)$$
$$= 0.00485043158777$$
$$\hat{a} = \frac{0.27000(0.893479 - 0.821797)^2 + 0.70986(0.561241 - 0.821797)^2 + 0.94609(1.146068 - 0.821797)^2 + 1(0.616774 - 0.821797)^2 - 4 \times 0.00}{0.27000 + 0.70986 + 0.94609 + 1 - \frac{0.27000^2 + 0.70986^2 + 0.94609^2 + 1^2}{0.27000 + 0.70986 + 0.94609 + 1}}$$
$$= 0.0825006309945$$

This gives the credibilities

Dev. Year j	β_j	Z_j
0	0.270002455704	$\frac{0.270002455704}{0.270002455704+\frac{0.00485043158777}{0.0825006309945}} = 0.82118754936$
1	0.709855576487	$\frac{0.709855576487}{0.709855576487+\frac{0.00485043158777}{0.0825006309945}} = 0.923511617335$
2	0.946091644205	$\frac{0.946091644205}{0.946091644205+\frac{0.00485043158777}{0.0825006309945}} = 0.941493105017$
3	1	$\frac{1}{1+\frac{0.00485043158777}{0.0825006309945}} = 0.944471979572$

The book premium is

$$\hat{\mu} = \frac{0.821187549360 \times 0.893479 + 0.923511617335 \times 0.561241 + 0.941493105017 \times 1.146068 + 0.944471979572 \times 0.616774}{0.821187549360 + 0.923511617335 + 0.941493105017 + 0.944471979572} = 0.8024887714$$

The credibility estimates for Ultimate losses and outstanding per-premium payments for each year are:

Acc. Year i	$\hat{C}_{i,J}$	$\hat{C}^{\text{BS}}_{i,J}$	$\hat{C}^{\text{BS}}_{i,J}(1-\hat{\beta}_{I-i})$
0	0.893479	$0.944472 \times 0.893479 + 0.055528 \times 0.802489 = 0.888426$	$0.888426 \times (1-1) = 0$
1	0.561241	$0.941493 \times 0.561241 + 0.058507 \times 0.802489 = 0.575356$	$0.575356 \times (1-0.946092) = 0.031016$
2	1.146068	$0.923512 \times 1.146068 + 0.076488 \times 0.802489 = 1.119788$	$1.119788 \times (1-0.709856) = 0.324900$
3	0.616774	$0.821188 \times 0.616774 + 0.178812 \times 0.802489 = 0.649982$	$0.649982 \times (1-0.270002) = 0.474485$

The total outstanding payments are

$$0.0310164775218 \times 6503 + 0.324900296088 \times 8152 + 0.474485346623 \times 7350 = \$6,338$$

19. *An insurance company collects the following run-off table for incremental losses.*

Accident year	*Development year* 0	1	2	3	4	5	6	7	8
2014	1027	942	403	264	374	143	67	24	11
2015	1096	1022	498	302	472	174	85	43	
2016	1109	1161	545	354	522	133	74		
2017	1153	1392	694	373	634	339			
2018	1336	1511	688	404	586				
2019	1280	1429	694	433					
2020	1449	1602	728						
2021	1702	1899							
2022	1693								

The earned premiums for each year were as follows:

Accident year	*Earned Premiums*
2014	4324
2015	4720
2016	4939
2017	5873
2018	6343
2019	6869
2020	7205
2021	7795
2022	7538

They have used the chain-ladder method to estimate claims reserves.

Use the Spearman's rank correlation to test whether the Development year 0 and 2 payments are correlated in different accident years.

The per-premium losses for Development years 0 and 2, Accident years 2014–2020 are

Accident year	Development year 0	2
2014	$\frac{1027}{4324} = 0.237511563367$	$\frac{403}{4324} = 0.093200740056$
2015	$\frac{1096}{4720} = 0.232203389831$	$\frac{498}{4720} = 0.105508474576$
2016	$\frac{1109}{4939} = 0.224539380441$	$\frac{545}{4939} = 0.110346223932$
2017	$\frac{1153}{5873} = 0.196322152222$	$\frac{694}{5873} = 0.11816788694$
2018	$\frac{1336}{6343} = 0.210625886804$	$\frac{688}{6343} = 0.10846602554$
2019	$\frac{1280}{6869} = 0.186344446062$	$\frac{694}{6869} = 0.101033629349$
2020	$\frac{1449}{7205} = 0.201110340042$	$\frac{728}{7205} = 0.101040943789$

The corresponding ranks are therefore:

Accident year	Development year 0	2	Squared difference of ranks
2014	7	1	36
2015	6	4	4
2016	5	6	1
2017	2	7	25
2018	4	5	1
2019	1	2	1
2020	3	3	0
Total			68

Spearman's correlation coefficient is therefore given by $r_s = 1 - \frac{6\times 68}{7\times 48} = -0.21428571429$. We form a test statistic $T = -0.21428571429\sqrt{\frac{5}{1-0.21428571429^2}} = -0.490552455136$, which is compared to a t-distribution with 5 degrees of freedom. This value is not significant, so there is not strong evidence that these development years are correlated.

20. *An insurance company has collected the following run-off table for incremental per-premium losses.*

Accident year	*Development year* 0	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*
2014	*0.1427*	*0.2342*	*0.2033*	*0.0714*	*0.0874*	*0.0243*	*0.0167*	*0.0164*	*0.0114*
2015	*0.1496*	*0.2422*	*0.1842*	*0.0702*	*0.0772*	*0.0224*	*0.0285*	*0.0143*	
2016	*0.1809*	*0.2261*	*0.1754*	*0.0854*	*0.0822*	*0.0283*	*0.0174*		
2017	*0.1753*	*0.2392*	*0.1793*	*0.0773*	*0.0734*	*0.0539*			
2018	*0.1536*	*0.2311*	*0.1808*	*0.0724*	*0.0686*				
2019	*0.1780*	*0.2429*	*0.1848*	*0.0633*					
2020	*0.1549*	*0.2602*	*0.1883*						
2021	*0.1702*	*0.2299*							
2022	*0.1693*								

Use the binomial test to determine whether Calendar year 2021 is unusual.

The ranks of payments in Calendar year 2021 are:

Accident year	Rank of 2021	above median
2014	2/2	Y
2015	3/3	Y
2016	3/4	Y
2017	2/5	N
2018	4/6	Y
2019	5/7	Y
2020	8/8	Y
2021	6/9	Y

Calendar year 2021 is above the median for 7 out of 8 years. The probability of this or a more extreme event is

$$2 \times \left(\frac{1}{2}\right)^8 \left(\binom{8}{0} + \binom{8}{1}\right) = \frac{9}{128} = 0.0703125$$

so Calendar year 2021 does not have a significant number of payments above the median.

21. *An insurance company uses a Poisson model for outstanding claims. They estimate the following parameters:*

Accident Year i	μ_i	*Dev. Year j*	γ_j
0	*1523*	*0*	*0.124*
1	*1952*	*1*	*0.382*
2	*2120*	*2*	*0.290*
3	*2084*	*3*	*0.147*
4	*2302*	*4*	*0.057*

It is currently the start of calendar year 5 (so development year 0 has just finished for accident year 4). Using these estimated values, what is the probability that the outstanding claims exceed 3,700?

Under the Poisson model, the outstanding claims follow a Poisson distribution with mean

$$\sum_{i+j>4} \mu_i\gamma_j = 1952\times0.057+2120(0.147+0.057)+2084(0.290+0.147+0.057)+2302(0.382+0.290+0.147+0.057) = 3589.792$$

This is approximated by a normal distribution with mean 3589.792 and variance 3589.792. The probability that outstanding claims exceed 3700 is therefore $1 - \Phi\left(\frac{3700-3589.792}{\sqrt{3589.792}}\right) = 1 - \Phi(1.83940972555) = 0.03292747$.

22. *An insurance company collects the following cumulative run-off triangle:*

	Development year			
Accident year	*0*	*1*	*2*	*3*
0	*5539*	*6003*	*6829*	*7108*
1	*6243*	*6792*	*7314*	
2	*6217*	*7209*		
3	*6372*			

and estimates the following reserves using the chain-ladder method:

Accident Year i	$\hat{C}_{i,J}$	*Dev. Year j*	f_j	γ_j	β_j
0	7108	*0*	*1.1113950775*	*0.782059819772*	*0.782059819772*
1	7612.81476056	*1*	*1.10535365377*	*0.087117614236*	*0.869177434008*
2	8294.04873842	*2*	*1.04085517645*	*0.091571018443*	*0.960748452451*
3	8147.71432939	*3*		*0.039251547549*	*1*

Under Mack's model, what is the mean squared error of the outstanding claims, including both process variance and squared estimation error?

We first estimate $\hat{\sigma}_j^2 = \frac{1}{I-1-j}\sum_{i=0}^{I-1-j} C_{ij}\left(f_{ij} - \hat{f}_j\right)^2$:

$$\hat{\sigma}_0^2 = \frac{1}{2}\left(5539\left(\frac{6243}{5539} - 1.1113950775\right)^2 + 6243\left(\frac{6217}{6243} - 1.1113950775\right)^2 + 6217\left(\frac{6372}{6217} - 1.1113950775\right)^2\right) = 65.6065698075$$

$$\hat{\sigma}_1^2 = \frac{1}{1}\left(6003\left(\frac{6792}{6003} - 1.10535365377\right)^2 + 6792\left(\frac{7209}{6792} - 1.10535365377\right)^2\right) = 17.2073933637$$

$$\hat{\sigma}_2^2 = \min\left(65.6065698075, 17.2073933637, \frac{65.6065698075^2}{17.2073933637}\right) = 17.2073933637$$

Then we estimate the process variance:

$$\mathrm{Var}(C_{i,J}|C_{i,I-i}) \approx \hat{C}_{i,J}^2 \sum_{j=I-i}^{J-1} \frac{\hat{\sigma}_j^2}{\hat{f}_j^{\,2}\hat{C}_{i,j}}$$

$$\mathrm{Var}(C_{1,J}|C_{1,3}) \approx 7612.81476056^2\left(\frac{17.2073933637^2}{1.04085517645^2 \times 7314}\right) = 2165634.34194$$

$$\mathrm{Var}(C_{2,J}|C_{2,2}) \approx 8294.04873842^2\left(\frac{17.2073933637^2}{1.04085517645^2 \times 7968.49448999} + \frac{17.2073933637^2}{1.10535365377^2 \times 7209}\right) = 4671948.16156$$

$$\mathrm{Var}(C_{3,J}|C_{3,1}) \approx 8147.71432939^2\left(\frac{17.2073933637^2}{1.04085517645^2 \times 7827.90393297 \times 7827.90393297} + \frac{17.2073933637^2}{1.10535365377^2 \times 7081.80943385}\right.$$
$$\left.+\frac{65.6065698075^2}{1.1113950775^2 \times 6372}\right) = 38575939.2472$$

so total process variance is $38575939.2472 + 4671948.16156 + 2165634.34194 = 45413521.7507$.

To compute estimation error, we first compute $S_j = \sum_{i=0}^{I-1-j} C_{i,j}$.

$$S_0 = 5539 + 6243 + 6217 + 6372 = 24371$$
$$S_1 = 6003 + 6792 + 7209 = 20004$$
$$S_2 = 6829 + 7314 = 14143$$
$$S_3 = 7108$$

The squared estimation error for each accident year is

$$\mathbb{E}\left(\left(\hat{C}_{i,J}-\mathbb{E}(C_{i,J}|D_I)\right)^2\right)\approx \hat{C}_{i,J}^2\sum_{j=I-i}^{J}\frac{\hat{\sigma}_j^2}{\hat{f}_j^{\,2}S_j}$$

$$\mathbb{E}\left(\left(\hat{C}_{1,J}-\mathbb{E}(C_{1,J}|D_I)\right)^2\right)\approx 7612.81476056^2\left(\frac{17.2073933637}{7108\times 1.04085517645}\right)=134793.169747$$

$$\mathbb{E}\left(\left(\hat{C}_{2,J}-\mathbb{E}(C_{2,J}|D_I)\right)^2\right)\approx 8294.04873842^2\left(\frac{17.2073933637}{14143\times 1.10535365377^2}+\frac{17.2073933637}{7108\times 1.04085517645}\right)=228498.667718$$

$$\mathbb{E}\left(\left(\hat{C}_{3,J}-\mathbb{E}(C_{3,J}|D_I)\right)^2\right)\approx 8147.71432939^2\left(\frac{65.6065698075}{20004\times 1.1113950775^2}+\frac{17.2073933637}{14143\times 1.10535365377^2}+\frac{17.2073933637}{7108\times 1.04085517645}\right)=396771.46$$

The total of MSEs for all years is $134793.169747+228498.667718+396771.46838=760063.305845$ and the cross-estimation error terms are given by

$$\mathbb{E}\left(\left(\hat{C}_{i,J}-\mathbb{E}(C_{i,j}|D_I)\right)\left(\hat{C}_{i',J}-\mathbb{E}(C_{i',j}|D_I)\right)\right)\approx \hat{C}_{i,J}\hat{C}_{i',J}\sum_{j=I-(i\wedge i')}^{J}\frac{\hat{\sigma}_j^2}{\hat{f}_j^{\,2}S_j}$$

So

$$\mathbb{E}\left(\left(\hat{C}_{1,J}-\mathbb{E}(C_{1,J}|D_I)\right)\left(\hat{C}_{2,J}-\mathbb{E}(C_{2,J}|D_I)\right)\right)\approx 7612.81476056\times 8294.04873842\frac{17.2073933637}{1.04085517645^2\times 7108}=141090.866414$$

$$\mathbb{E}\left(\left(\hat{C}_{1,J}-\mathbb{E}(C_{1,J}|D_I)\right)\left(\hat{C}_{3,J}-\mathbb{E}(C_{3,j}|D_I)\right)\right)\approx 7612.81476056\times 8147.71432939\frac{17.2073933637}{1.04085517645^2\times 7108}=138601.557609$$

$$\begin{aligned}\mathbb{E}\left(\left(\hat{C}_{2,J}-\mathbb{E}(C_{2,j}|D_I)\right)\left(\hat{C}_{3,J}-\mathbb{E}(C_{3,j}|D_I)\right)\right)&\approx 8294.04873842\times 8147.71432939\left(\frac{17.2073933637}{1.10535365377^2\times 14143}+\frac{17.2073933637}{1.04085517645^2\times 7108}\right)\\&=218297.887667\end{aligned}$$

The total squared estimation error due to cross-terms is $2(141090.866414+138601.557609+218297.887667)=995980.62338$.

Thus, the total MSE is $45413521.7507+760063.305845+995980.62338=47169565.6799$

23. *An actuary is reviewing the following incremental loss development triangles:*

No. of claims reported

Acc. Year	*0*	*1*	*2*	*3*
2019	*843*	*159*	*9*	*0*
2020	*862*	*164*	*11*	
2021	*830*	*166*		
2022	*844*			

No. of claims finalised

Acc. Year	*0*	*1*	*2*	*3*
2019	*428*	*338*	*186*	*59*
2020	*442*	*352*	*203*	
2021	*435*	*325*		
2022	*451*			

Payments (000's)

Acc. Year	*0*	*1*	*2*	*3*
2019	*211*	*200*	*144*	*71*
2020	*213*	*231*	*153*	
2021	*227*	*176*		
2022	*209*			

(Column headings 0–3 are *Development Year*.)

(a) Use the chain-ladder method to predict the numbers of claims settled in each year

The cumulative numbers of reported claims are:

Acc.	Development Year			
Year	0	1	2	3
2019	843	1002	1011	1011
2020	862	1026	1037	
2021	830	996		
2022	844			

Using the chain ladder method, the development factors for reported claims are

Development year j	f_j
0	$\frac{1002+1026+996}{843+862+830} = 1.19289940828$
1	$\frac{1011+1037}{1002+1026} = 1.00986193294$
2	$\frac{1011}{1011} = 1$

The estimated ultimate reported claims are therefore

Acc. Year	Ultimate Reported Claims
2019	1011
2020	$1037 \times 1 = 1037$
2021	$996 \times 1.00986193294 \times 1 = 1005.82248521$
2022	$844 \times 1.19289940828 \times 1.00986193294 \times 1 = 1016.7361647$

The cumulative numbers of settled claims are:

Acc.	Development Year			
Year	0	1	2	3
2019	428	766	952	1011
2020	442	794	997	
2021	435	760		
2022	451			

The development factors for settled claims are

Development year j	f_j	β_j	γ_j
0	$\frac{766+794+760}{428+442+435} = 1.77777777778$	0.42395628588	0.42395628588
1	$\frac{952+997}{766+794} = 1.24935897436$	0.75370006379	0.32974377791
2	$\frac{1011}{952} = 1.06197478992$	0.941641938671	0.187941874881
3		1	0.058358061329

Thus, the expected number of claims finalised are:

Acc.	Development Year			
Year	0	1	2	3
2020				60.5173095982
2021			189.036163668	58.697850278
2022		335.262424086	191.087301053	59.334751455

(b) Estimate the outstanding claims

We divide total losses by total number of claims settled to get average payments in each development year:

Dev. Year	Total claims finalised	Total payments made	Average payment per claim
0	1756	860	0.489749430524
1	1015	607	0.598029556650
2	389	297	0.763496143959
3	59	71	1.20338983051

We multiply the projected settled claims by these values to get the expected payments:

Acc.	Development Year			
Year	0	1	2	3
2020				$60.5173 \times 1.20339 = 72.826$
2021			$189.0362 \times 0.76350 = 144.328$	$58.6979 \times 1.20339 = 70.636$
2022		$335.2624 \times 0.59803 = 200.497$	$191.0873 \times 0.76350 = 145.894$	$59.3348 \times 1.20339 = 71.403$

Total outstanding claims are therefore $72.826 + 144.328 + 70.636 + 200.497 + 145.894 + 71.403 = 705.585$.

24. *An insurer classifies policies into three classes — single, couple and family. The experience from policy year 2016 is:*

Age Class	*Current differential*	*Earned premiums*	*Loss payments*
Single	*0.74*	*4,740*	*3,940*
Couple	*0.93*	*4,490*	*3,880*
Family	*1*	*5,670*	*4,930*

The base premium was \$420. Claim amounts are subject to 4% annual inflation. If the expense ratio is 25%, calculate the new premiums for each age class for policy year 2018. [15 mins]

Using the loss ratio method, the loss ratios are:

Class	loss ratio
Single	$\frac{3940}{4740} = 0.831223628692$
Couple	$\frac{3880}{4490} = 0.864142538976$
Family	$\frac{4930}{5670} = 0.869488536155$

The new differentials for couples should therefore be $0.93 \times \frac{0.864142538976}{0.869488536155} = 0.92428195178$. The new differential for singles should be $0.74 \times \frac{0.831223628692}{0.869488536155} = 0.70743369194$. Using these differentials, the total earned premiums in policy year 2016 would have been $5670 + 4490 \times \frac{0.92428195178}{0.93} + 4740 \times \frac{0.70743369194}{0.74} = 14663.7931034$, so the overall loss ratio would have been $\frac{12750}{14663.7931034} = 0.869488536158$. The target loss ratio is $1 - 0.25 = 0.75$, so the increase in base premium before inflation is $\frac{0.869488536158}{0.75} = 1.15931804821$. Two years of inflation is $(1.04)^2$, so the increase in base premium is $1.15931804821 \times (1.04)^2 = 1.25391840094$. The new base premium is $420 \times 1.25391840094 =$ \$526.645728395, and the new premium for a couple is $526.645728395 \times 0.92428195178 =$ \$486.77 and the new premium for a single policyholder is $526.645728395 \times 0.70743369194 =$ \$372.57.

25. *An insurer has different premiums for personal and commercial vehicles. Its experience for accident year 2016 is given below. There was a rate change on 1st August 2015, which affects some policies in 2016.*

Type	*Differential before rate change*	*Current differential*	*Earned premiums*	*Loss payments*
Personal	*1*	*1*	*11,300*	*9,800*
Commercial	*1.51*	*1.67*	*7,600*	*6,300*

Before the rate change, the base premium was \$950. The current base premium is \$1,020. Assuming that policies were sold uniformly over the year, calculate the new premimums for policy year 2018 assuming 6% annual inflation and a permissible loss ratio of 0.75. [15 mins]

The old premium applied for $\frac{7}{12}$ of 2015. Policies with this premium were therefore in force for $\frac{1}{2}\left(\frac{7}{12}\right)^2 = \frac{49}{288}$ of earned premium in 2016. Adjusting to the new premiums, the earned premium for personal in 2016 is $11300 \times \frac{1020}{1020\times\frac{239}{288}+950\times\frac{49}{288}} = 11433.4998106$. The adjusted earned premium for commercial policies in 2016 is $7600 \times \frac{1.67\times1020}{1.67\times1020\times\frac{239}{288}+1.51\times950\times\frac{49}{288}} = 7809.75640923$.

This means that the adjusted loss ratios are $\frac{9800}{11433.4998106} = 0.85713037673$ and $\frac{6300}{7809.75640923} = 0.80668329073$. The differential needs to be adjusted by a factor of $\frac{0.80668329073}{0.85713037673}$, so the new differential is $1.67 \times \frac{0.80668329073}{0.85713037673} = 1.57171082964$. Using this differential, total adjusted earned premiums in 2016 would be $11433.4998106+7809.75640923\times\frac{0.80668329073}{0.85713037673} = 18783.6068317$. The loss ratio is then $\frac{16100}{18783.6068317} = 0.85713037673$. The target loss ratio is 0.75, so without inflation, premiums need to be increased by a factor $\frac{0.85713037673}{0.75} = 1.14284050231$. Losses in accident year 2016 experience average inflation $\int_0^1 e^{\log(1.06)t}\,dt = \frac{0.06}{\log(1.06)} = 1.02970867194$ from the start of the year, while losses in policy year 2018 experience average inflation

$$\begin{aligned}\int_0^1 te^{\log(1.06)t}\,dt + 1.06\int_0^1 (1-t)e^{\log(1.06)t}\,dt &= 1.06\int_0^1 e^{\log(1.06)t}\,dt - 0.06\int_0^1 te^{\log(1.06)t}\,dt\\ &= 1.06\times\frac{0.06}{\log(1.06)} - 0.06\left(\frac{1.06}{\log(1.06)} - \frac{0.06}{\log(1.06)^2}\right)\\ &= 1.06029994908\end{aligned}$$

from the start of 2018. The base premium therefore needs to change by a factor $1.14284050231 \times (1.06)^2 \times \frac{1.06029994908}{1.02970867194} = 1.32224436298$. The new base premium is $1.32224436298 \times 1020 = \$1,348.69$, and the new premium for commercial policies is $1348.68925024 \times 1.57171082964 = \$2,119.75$.

26. *An insurance company has the following data for accident year 2017:*

		Earned Premiums		*Loss Payments*	
		House	*Appartment*	*House*	*Appartment*
Differential		*1*	*0.88*	*1*	*0.88*
Halifax	*1*	*5,200*	*4,100*	*4,150*	*3,600*
Dartmouth	*0.84*	*3,700*	*2,900*	*2,080*	*2,430*
Bedford	*1.25*	*4,400*	*2,500*	*3,820*	*2,030*

The base premium in 2017 was \$840. Calculate new premiums for policy year 2018 using inflation of 3% per year and expense ratio of 0.2.

We first calculate the new differentials. We obtain the following loss ratios:

Halifax	$\frac{7750}{9300} = 0.833333333333$
Dartmouth	$\frac{4510}{6600} = 0.683333333333$
Bedford	$\frac{5850}{6900} = 0.847826086957$
House	$\frac{10050}{13300} = 0.755639097744$
Apartment	$\frac{8060}{9500} = 0.848421052632$

The new differentials are therefore

Dartmouth	$0.84 \times \frac{0.683333333333}{0.833333333333} = 0.6888$
Bedford	$1.25 \times \frac{0.847826086957}{0.833333333333} = 1.27173913044$
Apartment	$0.88 \times \frac{0.848421052632}{0.755639097744} = 0.988051741292$

Balancing back the adjusted earned premiums are

$$5200+4100\times\frac{0.988051741292}{0.88}+3700\times\frac{0.6888}{0.84}+2900\times\frac{0.6888}{0.84}\times\frac{0.988051741292}{0.88}+4400\times\frac{1.27173913044}{1.25}+2500\times\frac{1.27173913044}{1.25}\times$$

The loss ratio is therefore $\frac{18110}{22839.7118581} = 0.792917183567$. To obtain an expense ratio of 0.2, the base premium therefore needs to be multiplied by $\frac{0.792917183567}{0.8} = 0.991146479459$.

The expected inflation from the start of 2017 to a random loss is $\int_0^1 (1.03)^t\,dt = \frac{0.03}{\log(1.03)} = 1.01492610407$. The expected inflation from the start of 2018 to a random loss in policy year 2018 is

$$\begin{aligned}\int_0^1 t(1.03)^t\,dt + \int_1^2 (2-t)(1.03)^t\,dt &= \int_0^1 t(1.03)^t\,dt + 1.03\int_0^1 (1-t)(1.03)^t\,dt\\ &= 1.03\int_0^1 (1.03)^t\,dt - 0.03\int_0^1 t(1.03)^t\,dt\\ &= 1.03\frac{0.03}{\log(1.03)} - 0.03\left(\frac{1.03}{\log(1.03)} - \frac{0.03}{\log(1.03)^2}\right)\\ &= \left(\frac{0.03}{\log(1.03)}\right)^2\\ &= 1.03007499672\end{aligned}$$

The new base premium is therefore $840 \times 0.991146479459 \times \frac{1.03\times1.03007499672}{1.01492610407} = 870.339664328$. The new premiums are therefore:

	House	Appartment
Halifax	\$870.34	\$859.94
Dartmouth	\$599.49	\$592.33
Bedford	\$1,106.85	\$1,093.62