ACSC/STAT 4703, Actuarial Models II

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Homework Sheet 1

Model Solutions

Basic Questions

1. *An insurance company models losses as following a Pareto distribution with $\alpha = 3.5$ and $\theta = 2000$. The fixed expenses are \$200 per claim, and variable expenses are 14% of loss amount. What is the density function of the distribution of the total cost to the insurance company for a random loss?*

   If the loss amount is X, then the insurer pays a total of $1.14X + 200$. $1.14X$ follows a Pareto distribution with $\alpha = 3.5$ and $\theta = 2280$. The density function is therefore

$$f(x) = \begin{cases} \frac{3.5 \times 2280^{3.5}}{(2080+x)^{4.5}} & \text{if } x > 200 \\ 0 & \text{otherwise} \end{cases}$$

2. *An insurer is modelling losses using a generalised regression model. Under their model, the losses X_i for a given policyholder should follow an inverse gamma distribution with shape $\alpha = 3$ and scale θ_i estimated by the regression model. To assess the model, they record the square $\left(X_i - \frac{\theta_i}{2}\right)^2$. What is the density function for the distribution of this statistic.*

   The density for $X_i - \frac{\theta}{2}$ is

$$g(x) = \begin{cases} \frac{\theta_i^3 e^{-\frac{\theta_i}{x+\frac{\theta_i}{2}}}}{2\left(x+\frac{\theta_i}{2}\right)} & \text{if } x > -\frac{\theta_i}{2} \\ 0 & \text{otherwise} \end{cases}$$

   The density for $\left(X_i - \frac{\theta}{2}\right)^2$ is therefore

$$f(x) = \begin{cases} \frac{1}{2\sqrt{x}} \frac{\theta_i^3 e^{-\frac{\theta_i}{\sqrt{x}+\frac{\theta_i}{2}}}}{2\left(\sqrt{x}+\frac{\theta_i}{2}\right)^4} & \text{if } x > \frac{\theta_i^2}{4} \\ \frac{1}{2\sqrt{x}} \left(\frac{\theta_i^3 e^{-\frac{\theta_i}{\sqrt{x}+\frac{\theta_i}{2}}}}{2\left(\sqrt{x}+\frac{\theta_i}{2}\right)^4} + \frac{\theta_i^3 e^{-\frac{\theta_i}{\frac{\theta_i}{2}-\sqrt{x}}}}{2\left(\frac{\theta_i}{2}-\sqrt{x}\right)^4} \right) & \text{if } 0 < x < \frac{\theta_i^2}{4} \\ 0 & \text{if } x \leqslant 0 \end{cases}$$

3. *An insurance company has the following data on its policies:*

Policy limit	*Losses Limited to*				
	50,000	*100,000*	*200,000*	*500,000*	*1,000,000*
50,000	*8,131,429*				
100,000	*10,833,728*	*15,096,434*			
200,000	*15,763,797*	*22,145,370*	*25440902*		
500,000	*30,126,054*	*46,654,553*	*58336196*	*72339459*	
1,000,000	*20,899,468*	*29,641,835*	*41482022*	*44513950*	*42764662*

Use this data to calculate the ILF from \$50,000 to \$1,000,000 using

(a) The direct ILF estimate.

The direct ILF estimate is $\frac{42764662}{20899468} = 2.04620816185$

(b) The incremental method.

The incremental ILF is

$$\frac{42764662}{44513950}\times\frac{44513950+72339459}{41482022+58336196}\times\frac{41482022+58336196+25440902}{29641835+46654553+22145370}\times\frac{29641835+46654553+22145370+15096434}{20899468+30126054+15763797+10833728} = 2.09315682301$$

4. *An insurance company charges a risk charge equal to the square of the average loss amount, divided by 100,000. It has the following data on a set of 4,407 claims from policies with limit \$1,000,000.*

Losses Limited to	*100,000*	*500,000*	*1,000,000*
Total claimed	*\$950,249*	*\$1,318,024*	*\$1,451,334*

Calculate the ILF from \$100,000 to \$1,000,000.

The pure premium for limit \$100,000 is $\frac{950249}{4407} = 215.622645791$. The risk charge is $\frac{215.622645791^2}{100000} = 0.464931253779$. Thus the total premium is $215.622645791 + 0.464931253779 = 216.087577045$. The pure premium for limit \$1,000,000 is $\frac{1451334}{4407} = 329.324710688$. The risk charge is $\frac{329.324710688^2}{100000} = 1.0845476507$. Thus the total premium is $329.324710688 + 1.0845476507 = 330.409258339$. The ILF is therefore $\frac{330.409258339}{216.087577045} = 1.52905253906$.

Standard Questions

5. *An insurer divides losses into two parts: property and medical. It models the property losses as following an exponential distribution with mean* Θ*, and the medical losses as following an exponential distribution with mean*

2Θ, *where Θ varies between individuals, following an inverse gamma distribution with $\theta = 500$ and $\alpha = 3$. What is the probability that a random claim exceeds \$50,000?*

For a given value $\Theta = \theta$, the total losses are the sum of an exponential distribution with mean θ and an exponential distribution with mean 2θ. The density of this sum is given by

$$f(x) = \int_0^x \frac{e^{-\frac{y}{\theta}}}{\theta} \frac{e^{-\frac{x-y}{2\theta}}}{2\theta}\,dy = \frac{e^{-\frac{x}{2\theta}}}{2\theta^2}\int_0^x e^{y\left(\frac{1}{2\theta}-\frac{1}{\theta}\right)}\,dy = \frac{e^{-\frac{x}{2\theta}}}{2\theta^2}\int_0^x e^{-\frac{y}{2\theta}}\,dy = \frac{e^{-\frac{x}{2\theta}}\left(1-e^{-\frac{x}{2\theta}}\right)}{\theta}$$

The probability that the losses exceed \$50,000 for a given value of θ is

$$\frac{1}{\theta}\int_{50000}^{\infty}\left(e^{-\frac{x}{2\theta}} - e^{-\frac{x}{\theta}}\right)dx = \frac{1}{\theta}\left[\theta e^{-\frac{x}{\theta}} - 2\theta e^{-\frac{x}{2\theta}}\right]_{50000}^{\infty} = 2e^{-\frac{25000}{\theta}} - e^{-\frac{50000}{\theta}}$$

The marginal probability that a loss exceeds \$50,000 is the expectation of this over the distribution of Θ, which is

$$\int_0^\infty \frac{500^3}{2\theta^4}e^{-\frac{500}{\theta}}\left(2e^{-\frac{25000}{\theta}} - e^{-\frac{50000}{\theta}}\right)d\theta = \int_0^\infty \frac{500^3 e^{-\frac{25500}{\theta}}}{\theta^4}\,d\theta - \int_0^\infty \frac{500^3 e^{-\frac{50500}{\theta}}}{2\theta^4}\,d\theta = \frac{2\times 500^3}{25500^3} - \frac{500^3}{50500^3} = 0.0000141065672049$$

Solution based on inverse exponential [An early version of the homework said ambiguously that Θ follows an inverse exponential distribution with $\theta = 500$ and $\alpha = 3$. If this was interpreted as inverse exponential with $\theta = 500$, then the marginal probability is calculated as follows:]

$$\int_0^\infty \frac{500}{\theta}e^{-\frac{500}{\theta}}\left(2e^{-\frac{25000}{\theta}} - e^{-\frac{50000}{\theta}}\right)d\theta = \int_0^\infty \frac{1000e^{-\frac{25500}{\theta}}}{\theta}\,d\theta - \int_0^\infty \frac{500e^{-\frac{50500}{\theta}}}{\theta}\,d\theta = \frac{1000}{25500} - \frac{500}{50500} = 0.0293146961755$$

6. *An insurance company's premiums include a risk charge proportional to the square of the expected claim. This results in a 20% loading for it's policies with limit \$500,000. A reinsurer offers reinsurance of \$500,000 over \$500,000 for a loading of 45%. The insurer calculates that this buying this reinsurance would not affect its premium (i.e. the premium with limit \$500,000 plus the reinsurance premium is equal to the premium with limit \$1,000,000). What is the ILF from \$500,000 to \$1,000,000? (It is not 1.)*

Let the risk charge be $\frac{\mathbb{E}(X\wedge l)^2}{c}$ for some constant c. We are given that $\frac{\mathbb{E}(X\wedge 500000)^2}{c} = 0.2\mathbb{E}(X\wedge 500000)$, so $c = \frac{\mathbb{E}(X\wedge 500000)}{0.2}$. Let $x = \mathbb{E}(X \wedge 500000)$, and $y = \mathbb{E}(X \wedge 1000000)$. The reinsurer's expected payment is

$y-x$, so the reinsurer's premium is $1.45(y-x)$. The insurer's premium for a policy with limit \$1,000,000 is $y + \frac{y^2}{c} = y + \frac{0.2y^2}{x}$. Since the premiums with and without reinsurance are equal, we have

$$\begin{aligned} y + \frac{0.2y^2}{x} &= 1.2x + 1.45(y-x) = 1.45y - 0.25x \\ -0.45\frac{y}{x} + 0.2\left(\frac{y}{x}\right)^2 &= -0.25 \\ 0.2\left(\frac{y}{x} - 1\right)\left(\frac{y}{x} - 1.25\right) &= 0 \\ \frac{y}{x} &= 1.25 \end{aligned}$$

so the ILF s 1.25.