ACSC/STAT 4703, Actuarial Models II

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Toby Kenney

Homework Sheet 5

Model Solutions

1. *A workers' compensation insurance company classifies companies as high, medium or low risk. Annual claims (in thousands) from high risk companies follow an inverse gamma distribution with $\alpha = 2.6$ and $\theta = 24$. Annual claims (in thousands) from medium risk companies follow a Pareto distribution with $\alpha = 8$ and $\theta = 42$. Annual claims (in thousands) from low risk companies follow a gamma distribution with $\alpha = 0.3$ and $\theta = 10$. 22% of companies are high risk, 48% are medium risk and 30% are low risk.*

   (a) Calculate the expectation and variance of the aggregate annual claims from a randomly chosen company.

   - For a high-risk company, the expected claim is $\frac{24}{1.6} = 15$. The variance is $\frac{24^2}{1.6^2 \times 0.6} = 375$
   - For a medium-risk company, the expected claim is $\frac{42}{7} = 6$. The variance is $\frac{42^2 \times 8}{7^2 \times 6} = 48$
   - For a low-risk company, the expected claim is $10 \times 0.3 = 3$. The variance is $10^2 \times 0.3 = 30$.

   The overall expected claim amount is

   $$0.22 \times 15 + 0.48 \times 6 + 0.3 \times 3 = 7.08$$

   For the variance, we can either calculate the raw moment, then subtract the square of the mean, or use the law of total variance.

   Calculating raw moments:

   The expected squared claim amount is

   $$0.22 \times (15^2 + 375) + 0.48 \times (6^2 + 48) + 0.3 \times (3^2 + 30) = 184.02$$

   The variance of the claim amount is therefore $184.02 - 7.08^2 = 133.8936$.

   Law of total variance:

   The expected conditional variance is

   $$0.22 \times 375 + 0.48 \times 48 + 0.3 \times 30 = 114.54$$

The variance of conditional expectation is

$$0.22 \times 15^2 + 0.48 \times 6^2 + 0.3 \times 3^2 - 7.08^2 = 19.3536$$

so the total variance is $114.54 + 19.3536 = 133.8936$.

(b) Given that a company's annual claims (in thousands) over the past 3 years were \$1.4, \$89.6 and \$1.2, what are the expectation and variance of the company's claims next year?

- The likelihood of these claims for a high-risk company is

$$\frac{24^{3\times 2.6} e^{-\frac{24}{1.4}-\frac{24}{89.6}-\frac{24}{1.2}}}{(1.4 \times 89.6 \times 1.2)^{3.6}\Gamma(2.6)^3} = 1.634153 \times 10^{-14}$$

- The likelihood of these claims for a medium-risk company is

$$\frac{8^3 \times 42^{24}}{(42+1.4)^9(42+89.6)^9(42+1.2)^9} = 1.371477 \times 10^{-7}$$

- The likelihood of these claims for a low-risk company is

$$\frac{(1.4 \times 89.6 \times 1.2)^{-0.7} e^{-\frac{1.4+89.6+1.2}{10}}}{10^{3\times 0.3}\Gamma(0.3)^3} = 1.392437 \times 10^{-08}$$

The posterior probabilites are therefore:

$$\frac{0.22 \times 1.634153 \times 10^{-14}}{0.22 \times 1.634153 \times 10^{-14} 2.459858 \times 10^{-12} + 0.48 \times 1.371477 \times 10^{-7} + 0.3 \times 1.392437 \times 10^{-08}} = 5.135307$$

$$\frac{0.48 \times 1.371477 \times 10^{-7}}{0.22 \times 1.634153 \times 10^{-14} + 0.48 \times 1.371477 \times 10^{-7} + 0.3 \times 1.392437 \times 10^{-08}} = 0.940331076032$$

and

$$\frac{0.3 \times 1.392437 \times 10^{-08}}{0.22 \times 1.634153 \times 10^{-14} + 0.48 \times 1.371477 \times 10^{-7} + 0.3 \times 1.392437 \times 10^{-08}} = 0.059668872615$$

This means that the expected aggregate claim is

$$5.13530735047 \times 10^{-8} \times 15 + 0.940331076032 \times 6 + 0.059668872615 \times 3 = 5.82099384434$$

The expected squared aggregate claim is

$$5.13530735047 \times 10^{-8} \times 600 + 0.940331076032 \times 84 + 0.059668872615 \times 39 = 81.3149272305$$

so the variance of aggregate claims is

$$81.3149272305 - 5.82099384434^2 = 47.4309578947$$

2. *An insurance company sets the book pure premium for its homeowner's insurance at \$830. The expected process variance is 13,241,000 and the variance of hypothetical means is 291,000. If a policyholder has aggregate claims of \$11,400 over the past 19 years, calculate the credibility premium for this policyholder's next year's insurance using the Bühlmann model.*

   The credibility of 19 years of experience is $Z = \frac{19}{19+\frac{13241000}{291000}} = 0.294565796484$. The credibility premium for this individual is therefore $0.294565796484 \times \frac{11400}{19} + 0.705434203516 \times 830 = \$762.25$.

3. *An insurance company has the following data on its group health insurance policy for a company.*

Year	*1*	*2*	*3*	*4*	*5*
Exposure	*5,021*	*7,425*	*7,591*	*7,309*	*7,393*
Aggregate claims	*\$1,084,200*	*\$1,909,400*	*\$2,996,200*	*\$972,800*	*\$1,664,000*

   The book premium is \$881 per unit of exposure. The variance of hypothetical means per unit of exposure is 24,951. The expected process variance per unit of exposure is 9,257,327,024. Using a Bühlmann-Straub model, calculate the credibility premium for Year 6 if the company has 8,815 units of exposure.

   The company has aggregate claims of \$8,626,600 from 34,739 units of exposure. The credibility of the company's exposure is $Z = \frac{34739}{34739+\frac{9257327024}{24951}} = 0.0856148008228$ Therefore the company's new premium per unit of exposure is $0.0856148008228 \times \frac{8626600}{34739} + 0.914385199177 \times 881 = 826.83374911$. The total premium for this company for 8,815 units of exposure is therefore $8815 \times 826.83374911 = \$7,288,539.50$.

Standard Questions

4. *An automobile insurer classifies drivers as "low-risk" and "high-risk". It estimates that 76% of drivers are low-risk. Annual claims from low-risk drivers are modelled as following a gamma distribution with $\alpha = 1.6$ and $\theta = 585$. Annual claims from high-risk drivers are modelled as following an inverse gamma distribution with $\alpha = 6$ and $\theta = 6205$. A driver has two year's experience, and has claimed a total of \$1,514 in the past two years. Her net premium for the coming year, using the Bayesian approach is \$994. What were her two claims in the previous years? [You may need to use numerical methods to solve this.]*

Let the claims be x and y. We are given that $x+y=1514$. The Bayesian net premium is $1241p_h+936(1-p_h)$ where p_h is the posterior probability that this policyholder is high risk. We therefore solve

$$\begin{aligned}1241p_h+936(1-p_h)&=994\\305p_h&=58\\p_h&=0.190163934426\end{aligned}$$

The likelihood of the claims x and y from the low-risk gamma distribution is $l_l=\frac{x^{0.6}y^{0.6}e^{-\frac{x+y}{585}}}{585^{3.2}\Gamma(1.6)^2}$. The likelihood from the high-risk inverse gamma distribution is $l_h=\frac{6205^{12}e^{-\frac{6205}{x}-\frac{6205}{y}}}{120x^7y^7}$. Solving the equation for the posterior probability that the individual is high risk gives

$$\begin{aligned}0.190163934426=p_h&=\frac{24l_h}{24l_h+76l_l}\\0.190163934426\left(24+76\frac{l_l}{l_h}\right)&=24\\76\frac{l_l}{l_h}&=102.206896552\\\frac{l_l}{l_h}&=1.34482758621\end{aligned}$$

We therefore need to solve

$$\begin{aligned}\frac{\left(\frac{x^{0.6}y^{0.6}e^{-\frac{x+y}{585}}}{585^{3.2}\Gamma(1.6)^2}\right)}{\left(\frac{6205^{12}e^{-\frac{6205}{x}-\frac{6205}{y}}}{120x^7y^7}\right)}&=1.34482758621\\x^{7.6}y^{7.6}e^{\frac{6205}{x}+\frac{6205}{y}-\frac{1514}{585}}&=1.34482758621\frac{6205^{12}585^{3.2}\Gamma(1.6)^2}{120}\\(xy)^{7.6}e^{\frac{6205\times1514}{xy}-2.58803418803}&=2.086842\times10^{52}\\7.6\log(xy)+\frac{9394370}{xy}&=123.0581\end{aligned}$$

Numerically, we solve $xy=365363.869$. Given $x+y=1514$, this gives $x=\frac{-1514+\sqrt{1514^2+4\times365363.869}}{2}=211.72$ so the losses are \$211.72 and \$1,301.28.

5. *An insurance company uses the Bühlmann-Straub model to calculate credibility. A new customer pays the book premium for 304 units of exposure, paying a total net premium of \$79,290 in its first year. It claims a total of \$24,828. In the second year, there has been 5% inflation, the customer has 253 units of exposure and pays a credibility premium of \$65,021. The*

customer claims a total of \$31,090 in the second year. There is 4% inflation from the second year to the third year. In the third year, the customer has 370 units of exposure. What is the total premium they pay for these?

In the first year, the book premium per unit of exposure is $\frac{79290}{304} = 260.822368421$, and the customer's average claim per unit of exposure is $\frac{24828}{304} = 81.6710526316$. In the second year, the customer pays $\frac{65021}{253} = 257$ per unit of exposure. Adjusted for 5% inflation, this is equivalent to $257(1.05)^{-1} = 244.761904762$ in the first year. The credibility of the first year's experience is therefore given by solving

$$\begin{aligned} 81.6710526316Z + 260.822368421(1-Z) &= 244.761904762 \\ 179.151315789Z &= 16.060463659 \\ Z &= 0.0896474781012 \end{aligned}$$

In the Bühlmann-Straub model, this credibility is given by $Z = \frac{m}{m+K}$ where $K = \frac{\text{EPV}}{\text{VHM}}$. We therefore get

$$\begin{aligned} \frac{304}{304+K} &= 0.0896474781012 \\ 304 &= 0.0896474781012(304+K) = 27.2528333428 + 0.0896474781012K \\ K &= \frac{304 - 27.2528333428}{0.0896474781012} = 3087.06025556 \end{aligned}$$

In the second year, the customer's inflation-adjusted aggregate claims is $31090(1.05)^{-1} = 29609.5238095$. Thus, the inflation-adjusted overall loss per unit of exposure is $\frac{24828+29609.5238095}{304+253} = 97.7334359237$. The credibility of 557 units of exposure is $\frac{557}{557+3087.06025556} = 0.152851479102$. Thus, the third-year premium without inflation is $0.152851479102 \times 97.7334359237 + 0.847148520898 \times 260.822368421 = 235.893983864$. Applying inflation, and multiplying by the 360 units of experience, the total premium in the third year is $235.893983864 \times 370 \times 1.05 \times 1.04 = \$95,310.61$.

6. *An insurance company is pricing a tenant's insurance policy for an individual. It has 4 years of past history for this individual, and the annual claims from year i are denoted X_i. It uses the formula $\hat{X}_5 = \alpha_0 + \sum_{i=1}^{4} \alpha_i X_i$. It makes the following assumptions about the losses each year:*

   - *The expected aggregate claims was \$322 in Year 1 and has been increasing by 4% inflation each year since then.*
   - *The coefficient of variation for aggregate claims is 2.7 in each year.*
   - *The correlation between losses in years i and j is 0.47 if $i \neq j$. (Recall $Corr(X,Y) = \frac{\text{Cov}(X,Y)}{\sqrt{\text{Var}(X)\,\text{Var}(Y)}}$)*

Find a set of equations which can determine the values of α_i for $i = 0, 1, \ldots, 5$. [You do not need to solve these equations.]

We use our standard equations:

$$\mathbb{E}(X_5) = \alpha_0 + \sum_{i=1}^{4} \alpha_i \mathbb{E}(X_i)$$
$$\mathrm{Cov}(X_5, X_j) = \sum_{i=1}^{4} \alpha_i \,\mathrm{Cov}(X_i, X_j)$$

From the first condition, we have $\mathbb{E}(X_i) = 322(1.04)^{i-1}$, $\mathrm{Var}(X_i) = \left(2.7 \times 322(1.04)^{i-1}\right)^2 = 755856.36 \times 1.04^{(2i-2)}$, and for $i \neq j$,

$$\mathrm{Cov}(X_i, X_j) = 0.47 \times 755856.36(1.04)^{i+j-2} = 355252.4892(1.04)^{(i+j-2)}$$

Substituting in the numbers given, these equations become:

$$\begin{aligned}
322(1.04)^4 &= \alpha_0 + 322\alpha_1 + 322(1.04)^1\alpha_2 + 322(1.04)^2\alpha_3 + 322(1.04)^3\alpha_4\\
355252.4892(1.04)^4 &= 755856.36\alpha_1 + 355252.4892(1.04)\alpha_2 + 355252.4892(1.04)^2\alpha_3 + 355252.4892(1.04)^3\alpha_4\\
355252.4892(1.04)^5 &= 355252.4892(1.04)\alpha_1 + 755856.36(1.04)^2\alpha_2 + 355252.4892(1.04)^3\alpha_3 + 355252.4892(1.04)^4\alpha_4\\
355252.4892(1.04)^6 &= 355252.4892(1.04)^2\alpha_1 + 355252.4892(1.04)^3\alpha_2 + 755856.36(1.04)^4\alpha_3 + 355252.4892(1.04)^5\alpha_4\\
355252.4892(1.04)^7 &= 355252.4892(1.04)^2\alpha_1 + 355252.4892(1.04)^4\alpha_2 + 355252.4892(1.04)^5\alpha_3 + 755856.36(1.04)^6\alpha_4
\end{aligned}$$